Graph Polynomials and categoricity

Why is the chromatic polynomial a polynomial?

Johann A. Makowsky

Faculty of Computer Science,
Technion - Israel Institute of Technology,
Haifa, Israel

http://www.cs.technion.ac.il/~janos

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Joint work with T. Kotek (Haifa) and B. Zilber (Oxford)

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Boris Zilber and JAM

BZ: What are you studying nowadays?

JAM: Graph polynomials.

BZ: Uh? What? Examples ....

JAM: Matching polynomial, chromatic polynomial, characteristic polynomial, ... (detailed definitions) ...

BZ: I know these! They all occur as growth polynomials in $\aleph_0$-categorical, ω-stable models!

JAM: ??? Let's see!

Overview

- The chromatic polynomial: G. Birkhoff 1912
- Parametrized Numeric graph invariants
- Coloring properties: A model theoretic view
- Graph polynomials
- Definability of numeric graph invariants
- $\aleph_0$-categorical ω-stable first order structures and the growth of their finite approximations
- If time permits: Complexity (and algebraic geometry)

References

- J.A. Makowsky and B. Zilber, *Polynomial invariants of graphs and totally categorical theories*, MODNET Preprint No. 21, 2006.
- T. Kotek, J.A. Makowsky and B. Zilber, *On Counting Generalized Colorings*, CSL 2008, 17th EACSL Annual Conference on Computer Science Logic, Lecture Notes in Computer Science vol. 5213 (2008) pp. 339-353,
- J.A. Makowsky, *From a Zoo to a Zoology: Towards a General Theory of Graph Polynomials*, Theory of Computing Systems, 43 (2008), pp. 542-562.

Graph Polynomial Project:

http://www.cs.technion.ac.il/∼janos/RESEARCH/gp-homepage.html

The Chromatic Polynomial

and

Its Variations

The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-vertex-coloring** is a map

$$c : V(G) \to [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings

Theorem: (G. Birkhoff, 1912)

$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.

Proof:

(i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.

(ii) For any edge $e = E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) + \chi(G/e, \lambda)$.

Interpretation of $\chi(G, \lambda)$ for $\lambda \notin \mathbb{N}$

What's the point in considering $\lambda \notin \mathbb{N}$?

Stanley, 1973 For simple graphs G, $|\,\chi(G, -1)\,|$ counts the number of acyclic orientations of G.

Stanley, 1973 There are also combinatorial interpretations of $\chi(G, -m)$ for each $m \in \mathbb{N}$, which are more complicated to state.

Open: What about $\chi(G, \lambda)$ for each $m \in \mathbb{R} - \mathbb{Z}$?

The Four Color Conjecture

Birkhoff wanted to prove the Four Color Conjecture using techniques from real or complex analysis.

Conjecture:(Birkhoff and Lewis) If G is planar then $\chi(G, \lambda) \neq 0$ for $\lambda \in [4, +\infty) \subseteq \mathbb{R}$.

This was not very successful. However, for real roots of χ we know:

Jackson, 1993 For simple graphs G we have $\chi(G, \lambda) \neq 0$ for $\lambda \in (-\infty, 0)$, $\lambda \in (0, 1)$ and $\lambda \in (1, \frac{32}{27})$.

Birkhoff and Lewis, 1946 For planar graphs G we have $\chi(G, \lambda) \neq 0$ for $\lambda \in [5, +\infty)$.

Still open: Are there planar graphs G such that $\chi(G, \lambda) = 0$ for some $\lambda \in (4, 5)$?

Thomassen, 1997 and Sokal, 2004 The real roots of all chromatic polynomials are dense in $(1, \frac{32}{27})$; the complex roots are dense in $\mathbb{C}$.

The edge-chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-edge-coloring** is a map

$$c : E(G) \to [\lambda]$$

such that if $(e, f) \in E(G)$ have a common vertex then $c(e) \neq c(f)$.

We define $\chi_e(G, \lambda)$ to be the number of λ- edge-colorings

Fact: $\chi_e(G, \lambda)$ a polynomial in $\mathbb{Z}[\lambda]$.

Let $L(G)$ be the **line graph** of G.
$V(L(G)) = E(G)$ and $(e, f) \in E(L(G))$ iff e and f have a common vertex.

Observation: $\chi_e(G, \lambda) = \chi(L(G), \lambda)$, where $L(G)$ is the line graph of G.

Conclusion: $\chi_e(G, \lambda)$ **is a polynomial** in $\mathbb{Z}[\lambda]$.

Variations on coloring, I

We can count other coloring functions.

- Total colorings

  $f_V : V \to [\lambda_V]$, $f_E : E \to [\lambda_E]$ and $f = f_V \cup f_E$,
  with f_V a proper vertex coloring and f_E a proper edge coloring.

- Connected components

  $f_V : V \to [\lambda_V]$, If $(u, v) \in E$ then $f_V(u) = f_V(v)$.

- Pre-coloring extensions

  Given graph $G = (V, E)$ and an equivalence relation R on V and $f_V : V \to [\lambda_V]$, we require that if $(u, v) \in R$ they have the same color, and if $(u, v) \in E - R$ they have different colors.

Fact: The corresponding counting functions are polynomials in λ.

Variations on coloring, II

Hypergraph colorings

Given a hypergraph $G = (V, E)$ with $E \subset \wp(V)$.

- If we require that if $u \in e$ for some $e \in E$ which is not a singleton, then there is $v \in E, u \neq v$ with $f(u) \neq f(v)$, we have a weak hypergraph coloring.
- If we require that for every $e \in E$, for every $u, v \in E, u \neq v$ we have $f(u) \neq f(v)$, we have a strong hypergraph coloring.

Given a hypergraph $G = (V, E, D)$ with two types of hyper-edges $D, E \subset \wp(V)$.

- If we require that
  - if $u \in e$ for some $e \in E$, which is not a singleton, then there is $v \in E, u \neq v$ with $f(u) \neq f(v)$;
  - if $u, v \in d$ for some $d \in D$, then $f(u) = f(v)$;

  we have a mixed hypergraph coloring.

Fact: The corresponding counting functions are polynomials in λ.

Vitaly I. Voloshin, Coloring Mixed Hypergraphs: Theory, Algorithms and Applications,

Fields Institute Monographs, AMS 2002

Variations on coloring, III

Encountered at CanaDam-07:

Let $f : V(G) \to [\lambda]$ be a function, such that Φ is one of the properties below and $\chi_\Phi(G, \lambda)$ denotes the number of such colorings with atmost λ colors.

* **convex:** Every monochromatic set induces a connected graph.

* **injective:** f is injectiv on the neighborhood of every vertex.

- **complete:** f is a proper coloring such that every pair of colors occurs along some edge.

* **harmonious:** f is a proper coloring such that every pair of colors occurs at most once along some edge.

- **equitable:** All color classes have (almost) the same size.

* **equitable, modified:** All non-empty color classes have the same size.

New fact: For (*), $\chi_\Phi(G, \lambda)$ is a polynomial in λ, for (-), it is not.

Variations on coloring, IV

* **path-rainbow:** Let $f : E \rightarrow [\lambda]$ be an edge-coloring. f is **path-rainbow** if between any two vertices $u, v \in V$ there as a path where all the edges have different colors.

New fact: $\chi_{rainbow}(G, \lambda)$, the number of path-rainbow colorings of G with λ colors, is a polynomial in λ

Rainbow colorings of various kinds arise in computational biology

* **-monochromatic components:** Let $f : V \rightarrow [\lambda]$ be an vertex-coloring and $t \in \mathbb{N}$. f is an mcc_t-coloring of G with λ colors, if all the connected components of a monochromatic set have size at most t.

New fact: For fixed $t \geq 1$ the function $\chi_{mcc_t}(G, \lambda)$, the number of mcc_t-colorings of G with λ colors, is a polynomial in λ. but not in t.

mcc_t colorings were first studied in:

N. Alon, G. Ding, B. Oporowski, and D. Vertigan. Partitioning into graphs with only small components. *Journal of Combinatorial Theory, Series B*, 87:231–243, 2003.

Parametrized Numeric Graph Invariants

Bounded numeric invariants

In graph theory it is often customary to look at numeric invariants which bounded by a function $b : G \to \mathbb{N}$.

- $k(G)$: the number of connected components of G;
  $k(G, \lambda)$: the number of connected components of G of size λ.
- $cl(G)$: the number of cliques of G;
  $cl(G, \lambda)$: the number of cliques of G of size λ.
- $indep(G, \lambda)$: the number of independent sets of G of size λ.
- $v(G, \lambda)$: the number of vertex covers of G of size λ.
- $m(G, \lambda)$: the number of matchings of G of size λ.

Obviously, these functions are not polynomials in λ,
because they vanish for large enough λ.

Pngi's: Parametrized numeric graph invariants

Let $\mathcal{K}$ denote a class of finite (colored) graphs
(hypergraphs, or structures over some fixed vocabulary).

A **parametrized numeric graph invariant (pngi)** is a function $\alpha(G, \lambda)$

$$\mathcal{K} \times \mathbb{N} \rightarrow \mathbb{N}$$

such that, for each $\lambda \in \mathbb{N}$ and G_1 isomorphic to G_2
we have that $\alpha(G_1, \lambda) = \alpha(G_2, \lambda)$.

Let $\alpha(G, \lambda)$ and Let $\beta(G, \lambda)$ be two pngi's.
Clearly, we can form new such invariants by forming

- $\alpha(G, \lambda) + \beta(G, \lambda)$, $\alpha(G, \lambda) \cdot \beta(G, \lambda)$, $2^{\alpha(G,\lambda)}$
- If $\alpha(G, \lambda) = 0$ for all large enough λ,
$$\beta(G, \lambda) = \sum_n \alpha(G, n)\lambda^n$$

If $\alpha(G, \lambda) \in \mathbb{Z}[\lambda]$ is a polynomial, we speak of **graph polynomials**.

The behaviour of parametrized numeric graph invariants

The pngi's of the form $\alpha(G, \lambda)$ we have seen so far show the following behaviour:

- For each graph there is $b_G \in \mathbb{N}$ such that $\alpha(G, \lambda) \leq \lambda^{b_G}$.
- For each $n \in \mathbb{N}$ we have $\alpha(G, n) \in \mathbb{N}$.
- There is $n_G \in \mathbb{N}$ such that
  either $\alpha(G, n) = 0$ for all $n \geq n_G$
  or $\alpha(G, n)$ is not decreasing for all $n \geq n_G$.

Coloring Properties

A Model-Theoretic View

Enter logic: Model theory

Our framework is as follows:

- Let $\mathfrak{M}$ be a finite τ-structure with universe M.
- Let $k \in \mathbb{N}$ and $[k] = \{0, \ldots, k-1\}$.
- Let $\mathfrak{M}_k$ be the two-sorted τ' structure $\langle \mathfrak{M}, [k] \rangle$.
- Let F be an r-ary function symbol with interpretations in $\mathfrak{M}_k$ of the form $f : M^r \to [k]$.

Coloring properties, I

We denote *relation symbols* *by* **bold-face letters***, and their* *interpretation* *by the corresponding* roman-face letter.

Let $\tau_R = \tau_1 \cup \{\mathbf{R}\}$, where is $\mathbf{R}$ is a two-sorted relation symbol of arity $r = s + t$.

A class of τ_R- structures $\mathcal{P}$ is a **coloring property** if

Extension Property: Let $\mathcal{M}$ be fixed. Then $\mathcal{M}_k$ is a substructure of $\mathcal{M}_n$ for each $n \geq k$. Let R_0 be a fixed relation on $\mathcal{M}_k$. If $\langle \mathcal{M}_k, R_0 \rangle \in \mathcal{P}$ and $n \geq k$ then also $\langle \mathcal{M}_n, R_0 \rangle \in \mathcal{P}$.

Isomorphism Property: $\mathcal{P}$ is closed under τ_R-isomorphisms.

This implies the permutation property:

Permutation Property: Let $R \subseteq M^s \times [k]^t$ be a fixed relation on $\mathcal{M}_k$. For π is a permutation of $[k]$, We define $R_\pi = \{(\bar{m}, \pi(\bar{a})) \in M^\times [k]^t : (\bar{m}, \bar{a}) \in R\}$.

Then $\langle \mathcal{M}_k, R \rangle \in \mathcal{P}$ iff $\langle \mathcal{M}_k, R_\pi \rangle \in \mathcal{P}$.

We refer to $\mathbf{R}$ and its interpretations R as **coloring predicates**.

Coloring properties, II

(i) A coloring property is **bounded**, if for every $\mathcal{M}$ there is a number N_M such that for all $k \in \mathbb{N}$ the set of colors

$$\{x \in [k] : \exists \bar{y} \in M^m R(\bar{y}, x)\}$$

has size at most N_M.

(ii) A coloring property is **range bounded**, if its range is bounded in the following sense: There is a number $d \in \mathbb{N}$ such that for every $\mathcal{M}$ and $\bar{y} \in M^m$ the set $\{x \in [k] : R(\bar{y}, x)\}$ has at most d elements.

Clearly, if a coloring property is range bounded, it is also bounded.

Coloring properties, III

Let ϕ be a sentence of some logic $\mathcal{L}$.

$\mathcal{L}$ could be first order logic, second order logic, infinitary logic, or some fragment thereof.

(i) $\phi(\mathbf{R})$ is a **coloring formula**, if the class of its models, which are of the form of the form $\langle \mathcal{M}, [k], R \rangle$, is a **coloring property**.

(ii) Let $\mathcal{P}$ be a bounded coloring property. A relation $R_M \subset M^m \times [k]$ is a **generalised $k - \mathcal{P}$-coloring** if $\langle \mathcal{M}_k, R \rangle \in \mathcal{P}$.

(iii) We denote by

$$\chi_{\mathcal{P}}(\mathcal{M}, k)$$

the number of generalised $k - \mathcal{P}$-coloring R on $\mathcal{M}$.

If $\mathcal{P}$ is defined by $\phi(\mathbf{R})$ we also write

$$\chi_{\phi(R)}(\mathcal{M}, k).$$

Generalized multi-colorings, I

To construct also graph polynomials in several variables, we extend the definition to deal with several color-sets, and also call them generalized chromatic polynomials.

Let $\mathcal{M}$ be a τ-structure with universe M.

We say an $(\alpha + 1)$-sorted structure

$$\langle \mathcal{M}, [k_1], \ldots, [k_\alpha], R \rangle$$

for the vocabulary $\tau_{\alpha,R}$ with

$$R \subset M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$$

is a **generalized coloring** of $\mathcal{M}$ for colors $\bar{k}^\alpha = (k_1, \ldots, k_\alpha)$.

By abuse of notation,

$m_i = 0$ is taken to mean the color-set k_i is not used in R.

Generalized multi-colorings, II

A class of generalized multi-colorings $\mathcal{P}$ is a coloring property if it satisfies the following conditions:

Isomorphism property : $\mathcal{P}$ is closed under $\tau_{\alpha,R}$-isomorphisms.

Extension property : For every $\mathcal{M}$, $k_1 \leq k'_1, \ldots, k_\alpha \leq k'_\alpha$, and R,
if $\langle \mathcal{M}, [k_1], \ldots, [k_\alpha], R \rangle \in \mathcal{P}$ then $\langle \mathcal{M}, [k'_1], \ldots, [k'_\alpha], R \rangle \in \mathcal{P}$.

Non-occurrence property : Assume

$$R \subset M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$$

with $m_i = 0$, and

$$\langle \mathcal{M}, [k_1], \ldots, [k_\alpha], R \rangle \in \mathcal{P},$$

then for every $k'_i \in \mathbb{N}$,

$$\langle \mathcal{M}, [k_1], \ldots, [k'_i], \ldots, [k_\alpha], R \rangle \in \mathcal{P}.$$

The **boundedness conditions** are the obvious adaptions.

Main result, A

Generalized chromatic polynomials

Main result, A

THEOREM A: If $\phi(\mathbf{R})$ is an $\mathcal{L}$-sentence and defines a bounded coloring propert then

$$\chi_\phi(\mathfrak{M}, k_1, \ldots k_\alpha) \in \mathbb{Z}[k_1, \ldots k_\alpha]$$

is indeed a polynomial in $k_1, \ldots k_\alpha$.

We shall call polynomials obtained like this $\mathcal{L} - MG$-polynomials.

MG-polynomial for model theoretic growth polynomial

(as studied by B. Zilber in his work on categoricity).

Corollary: Taking $\mathcal{L}$ to be (monadic) second order logic,
this covers **all** the previous examples,
and allows us to construct **infinitely many more** MG-polynomials.

A theorem with an elementary generic proof

suggested simplification by A. Blass

We prove something a bit stronger (for the case of $\alpha = 1$, i.e., **one** color set):

THEOREM A': For every $\mathcal{M}$ the number $\chi_{\phi(R)}(\mathcal{M}, k)$ is a polynomial in k of the form

$$\sum_{j=0}^{d \cdot |M|^m} c_{\phi(R)}(\mathcal{M}, j) \binom{k}{j}$$

where $c_{\phi(R)}(\mathcal{M}, j)$ is the number of generalised $k - \phi$-colorings R with a fixed set of j colors.

In the light of this theorem we call $\chi_{\phi(R)}(\mathcal{M}, k)$

also a *generalised chromatic polynomial*.

Proof

We first observe that any generalised coloring R uses at most

$$N = d \cdot \mid M \mid^m$$

of the k colors.

For any $j \leq N$, let $c_{\phi(R)}(\mathcal{M}, j)$ be the number of colorings, with a fixed set of j colors, which are generalised vertex colorings and use all j of the colors.

Next we observe that any permutation of the set of colors used is also a coloring.

Therefore, given k colors, the number of vertex colorings that use exactly j of the k colors is the product of $c_{\phi(R)}(\mathcal{M}, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi_{\phi(R)}(\mathcal{M}, k) = \sum_{j \leq N} c_{\phi(R)}(\mathcal{M}, j) \binom{k}{j}$$

The right side here is a polynomial in k, because each of the binomial coefficients is. We also use that for $k \leq j$ we have $\binom{k}{j} = 0$. Q.E.D.

Graph polynomials

Prominent graph polynomials

- The chromatic polynomial (G. Birkhoff, 1912)
- The Tutte polynomial and its colored versions
  (W.T. Tutte 1954, B. Bollobas and O. Riordan, 1999);
- The characteristic polynomial
  (T.H. Wei 1952, L.M. Lihtenbaum 1956, L. Collatz and U. Sinogowitz 1957)
- The various matching polynomials (O.J. Heilman and E.J. Lieb, 1972)
- Various clique and independent set polynomials (I. Gutman and F. Harary 1983)
- The Farrel polynomials (E.J. Farrell, 1979)
- The cover polynomials for digraphs (F.R.K. Chung and R.L. Graham, 1995)
- The interlace-polynomials
  (M. Las Vergnas, 1983, R. Arratia, B. Bollobás and G. Sorkin, 2000)
- The various knot polynomials (of signed graphs)
  (Alexander polynomial, Jones polynomial, HOMFLY-PT polynomial, etc)

Application of graph polynomials

There are plenty of applications of graph polynomials in

- Graph theory proper
- Knot theory
- Chemistry
- Statistical mechanics
- Quantum physics
- Quantum computing
- Biology

Using our framework: The matching polynomial

We want to show that the matching polynomial can be obtained in our framework.

- For a graph $G = (V, E)$ we form a 4-sorted structure

$$\mathfrak{M}(G) = \langle V, E, \wp(V), \wp(E), \in, R_G \rangle$$

  where $\in$ is the membership relation between elements of V and $\wp(V)$, and elements of E and $\wp(E)$ respectively,
  and R_G is the incidence relation between vertices and edges.

- $\mathfrak{M}(G)_k = \langle V, E, \wp(V), \wp(E), \in, R_G, [k] \rangle$

- The formula $\phi_{matching}(m, f)$ now says:

  (i) $m \in \wp(E)$ is a matching.

  (ii) f is a function $f : m \to [k]$.

Using our framework: The matching polynomial, contd

We replace k by λ.

Now we put $\bar{g}(G, \lambda)$ to be the number of pairs (m, f) such that

$$\langle \mathfrak{M}(G)_\lambda, m, f \rangle \models \phi_{matching}(m, f)$$

- For fixed m there are $\lambda^{|m|}$ many f's satisfying the formula $\phi_{matching}(m, f)$.
- For matchings m with $| m |= j$ we get $m(G, j)\lambda^j$ many such pairs.
- Hence we get

$$\bar{g}(G, \lambda) = \sum_j m(G, j)\lambda^j = \sum_{\substack{M: M \subseteq E \\ M \text{ is a matching}}} \prod_{e: e \in M} \lambda = g(G, \lambda)$$

Definability of graph polynomials

in (Monadic) Second Order Logic SOL (MSOL)

Simple (M)SOL-definable graph polynomials

The graph polynomial $ind(G,X) = \sum_i ind(G,i) \cdot X^i$, can be written also as

$$ind(G,X) = \sum_{I \subseteq V(G)} \prod_{v \in I} X$$

where I ranges over all independent sets of G.
To be an independent set is definable by a formula of Monadic Second Order Logic (MSOL) $\phi(I)$.

A **simple MSOL-definable graph polynomial** $p(G,X)$ is a polynomial of the form

$$p(G,X) = \sum_{A \subseteq V(G) : \phi(A)} \prod_{v \in A} X$$

where A ranges over all subsets of $V(G)$ satisfying $\phi(A)$

and $\phi(A)$ is a (M)SOL-formula.

General (M)SOL-definable graph polynomials

For the general case

- One allows several indeterminates $X_1, \ldots, X_t$.
- One gives an inductive definition.
- One allows an ordering of the vertices.
- One requires the definition to be **invariant under the ordering**, i.e., different orderings still give the same polynomial.
- This also allows to define the modular counting quantifiers $C_{m,q}$ "there are, modulo q exactly m elements..."

The general case includes the Tutte polynomial, the cover polynomial, and virtually all graph polynomials from the literature.

Graph polynomials which are not $\mathbf{MSOL}$-definable

without the assumption $\mathbf{P} \neq \mathbf{NP}$

Let $c : E \rightarrow [k]$ be an edge-coloring. c is **path-rainbow** if between any two vertices $u, v \in V$ there as a path where all the edges have different colors.

We denote by $\chi_{rainbow}(G, k)$ the number of path-rainbow colorings of G with k colors.

Theorem:(T. Kotek and J.A.M.)

(i) $\chi_{rainbow}(G, k)$ is a polynomial in k.

(ii) $\chi_{rainbow}(G, k)$ is not $\mathbf{MSOL}$-definable (but $\mathbf{SOL}$-definable).

Proof: A more sophisticated use of connection matrices.

The same works also for harmonious colorings.

Main Result, B

All graph polynomials are
generalized chromatic polynomials

MG-polynomials and $\mathbf{SOL}$-polynomials

The definition of MG-polynomials is very flexible and
can be extended to multivariate polynomials.

THEOREM B: The extended $\mathbf{SOL}$-graph polynomials are exactly the SOL-definable MG-polynomials.

Remark: In the proof for the matching polynomial we we used the powersets of V and E as part of the structure $\mathfrak{M}(G)$. One can iterate this idea, hence also graph polynomials defined with higher order logic are MG-polynomials.

Remark: The theorem fails if we replace SOL by MSOL.

Zilber's growth functions

in

$\aleph_0$-categorical and ω-stable structures

The functor $\mathbb{M}$.

Let $\mathcal{G}$ be a class of finite structures of a finite language τ_0.

Let $D_1, \ldots, D_k$ be countable infinite structures of finite languages $\tau_1, \ldots, \tau_k$, correspondingly.

For every $G \in \mathcal{G}$ we construct the structure $\mathbb{M}(G, D_1, \ldots, D_k)$ of sorts $G, D_1, \ldots, D_k$ and F and the language $\tau = \tau_0 \cup \tau_1 \cdots \cup \tau_k$ and extra function symbol

$$\Phi : G \times F \to D_1 \times \ldots \times D_k.$$

The only condition on Φ is

$$\forall f, f' \in F([\forall g \in G\ \Phi(g, f) = \Phi(g, f')] \to f = f').$$

We identify elements $f \in F$ with functions $f : G \to D_1 \times \cdots \times D_k$ and write $f(g)$ instead of $\Phi(g, f)$.

In other words we have the canonical identification

$$\Phi^\star : F \leftrightarrow (D_1 \times \cdots \times D_k)^G,$$

and fixing an enumeration of G we may identify the right-hand-side with the cartesian power

$$(D_1 \times \cdots \times D_k)^{|G|}.$$

Invariants of G

- By the virtue of the construction, given $D_1, \dots, D_k$, the isomorphism type of $\mathbb{M}(G, D_1, \dots, D_k)$ depends only on G.
- Obviously, G can be recovered from $\mathbb{M}(G, D_1, \dots, D_k)$.
- So, $\mathbb{M}(G, D_1, \dots, D_k)$ can be seen as **the complete invariant of G**.

  In particular, every definable subset S of F is an invariant of G.

Observations

(i) $\mathbb{M}(G, D_1, \ldots, D_k)$ is definable using parameters in the disjoint union $D_1 \cup \cdots \cup D_k$.

(ii) Assume that the theory of each D_i is $\aleph_0$-categorical. Then the theory Th$[\mathbb{M}(G, D_1, \ldots, D_k)]$ is $\aleph_0$-categorical.

(iii) Assume that the theory of each D_i is strongly minimal. Then the theory Th$[\mathbb{M}(G, D_1, \ldots, D_k)]$ is ω-stable with k independent dimensions.
If $k = 1$ then the theory is categorical in uncountable cardinals.

Finite model property

Using Theorem 7 of G. Cherlin and E. Hrushovski, *Finite structures with few types*, we have

- Any $\aleph_0$-categorical and ω-stable theory has the finite model property. Moreover any countable model M can be represented as

$$M = \bigcup_{i=1}^{\infty} M_i,$$

  i.e., as a union of an increasing chain of finite substructures *(logically) approximating* M.

- The finite model property takes a very simple form for a strongly minimal structure D, namely,
  D has the finite model property if and only if acl(X) is finite for any finite $X \subset D$.

$\aleph_0$-categorical and ω-stable structures and the growth polynomials of their finite approximations

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Zilber's Theorem: Let $M = \mathbb{M}(G, D_1, \ldots, D_k)$.

Assume the finite model property holds in the strongly minimal structures $D_1, \ldots D_k$.

Then for every finite $C \subset M$ and any C-definable set $S \subset M^\ell$ a there is a polynomial $p_S \in \mathbb{Q}[x]$ and a number n_S such that for every finite $X \subseteq M$ with $C \subseteq X$,

(i) letting $|D_i \cap \text{acl}\,(X)| = x_i \geq n_S$, we have

$$|S \cap \text{acl}\, X| = p_S(x_1, \ldots, x_k);$$

(ii) $\text{rk}\,(S) = \deg(p_S)$, the degree of the polynomial;

(iii) if g(S)=T for some automorphism g of M then $p_S = p_T$ and $n_S = n_T$.

Furthermore, if $C = \emptyset$ we can take $n_S = 0$.

$\aleph_0$-categorical and ω-stable structures and graph polynomials

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THEOREM C: Fix $n \in \mathbb{N}$. There is a functor $\mathbb{M}_n$, mapping graphs G into infinite structures $\mathbb{M}_n(G)$, such that

(i) $\mathbb{M}_n(G)$ is $\aleph_0$-categorical and ω-stable;

(ii) every $\mathbf{SOL}$-definable graph polynomial P in n indeterminates occurs in $\mathbb{M}_n(G)$ as the growth function of a first order definable n-ary relation.

Remarks: It works for τ-structure rather than graphs.

It also works for graph polynomials definable in higher order logic.

Thank you for your attention !

If time would permit we could now discuss also **complexity** . . .

(only 20 minutes more)

Complexity of evaluations

References for Complexity, I

- L.G. Valiant,
  The Complexity of Enumeration and Reliability Problems,
  SIAM Journal on Computing, 8 (1979) 410-421

- N. Linial,
  Hard enumeration problems in geometry and combinatorics,
  SIAM Journal of Algebraic and Discrete Methods, 7 (1986), pp. 331-335.

- F. Jaeger, D.L. Vertigan, D.J.A. Welsh,
  On the computational complexity of the Jones and Tutte polynomials,
  Math. Proc. Cambridge Philos. Soc., 108 (1990) pp. 35-53.

References for Complexity, II

- Markus Bläser, Holger Dell,
  The complexity of the cover polynomial.
  ICALP'07, pp. 801-812

- Markus Bläser, Christian Hoffmann,
  On the Complexity of the Interlace Polynomial,
  STACS'08, pp. 97-108

- Markus Bläser, Holger Dell, J.A. Makowsky,
  Complexity of the Bollobas-Riordan Polynomial:
  Exceptional points and uniform reductions,
  CSR'08, pp. 86-98

The complexity of the chromatic polynomial, I

Theorem:

- $\chi(G, 3)$ is $\sharp\mathbf{P}$-complete (Valiant 1979).
- $\chi(G, -1)$ is $\sharp\mathbf{P}$-complete (Linial 1986).

Question: What is the complexity of computing $\chi(G, \lambda)$ for $\lambda = \lambda_0 \in \mathbb{Q}$ or even for $\lambda = \lambda_0 \in \mathbb{C}$?

The complexity of the chromatic polynomial, $\mathbf{II}$

Let $G_1 \bowtie G_2$ denote the join of two graphs.

We observe that

$$\chi(G \bowtie K_n, \lambda) = (\lambda)^{\underline{n}} \cdot \chi(G, \lambda - n) \qquad (\star)$$

Hence we get

(i) $\chi(G \bowtie K_1, 4) = 4 \cdot \chi(G, 3)$

(ii) $\chi(G \bowtie K_n, 3 + n) = (n + 3)^{\underline{n}} \cdot \chi(G, 3)$ hence for $n \in \mathbb{N}$ with $n \geq 3$ it is $\sharp\mathbf{P}$-complete.

The complexity of the chromatic polynomial, III

If we have have an oracle for some $q \in \mathbb{Q} - \mathbb{N}$ which allows us to compute $\chi(G, q)$ we can compute $\chi(G, q')$ for any $q' \in \mathbb{Q}$ as follows:

Algorithm $A(q, q', \mid V(G) \mid)$**:**

(i) Given G the degree of $\chi(G, q)$ is at most $n = \mid V(G) \mid$.

(ii) Use the oracle and $(\star)$ to compute $n + 1$ values of $\chi(G, \lambda)$.

(iii) Using Lagrange interpolation we can compute $\chi(G, q')$ in polynomial time.

We note that this algorithm is purely algebraic and works for all graphs G, $q \in (F) - \mathbb{N}$ and $q' \in F$ for any field F extending $\mathbb{Q}$.

The complexity of the chromatic polynomial, IV

We summarize the situation for the chromatic polynomial as follows:

(i) We have an **exception set** $C = \mathbb{N}$ which is a countable union of semi-algebraic sets of dimension 0 in the field $\mathbb{C}$.

(ii) We have a numeric graph invariant $f(G) =\mid G \mid$ which is $\mathbf{FP}$.

(iii) We have **one algebraic** algorithm $A(q, q', f(G))$ which runs in polynomial time in q, q' and $f(G)$ which calls the oracle $\chi(-, q')$.
q, q' are in any finite dimensional algebraic extension field F of $\mathbb{Q}$.

(iv) The algorithm $A(q, q', f(G))$ reduces **uniformly**,
for any $q \in F - C$, the evaluation of $\chi(G, q)$ into the evaluation of $\chi(G, 3)$.

The nature of the algorithm A, I

In the case of $\chi(-,q)$ and $\chi(-,q')$

- The input of A is $f(G)) \in F$,
  in this case the degree of the $\chi(G,\lambda)$
- The output of A is a rational function $A(q,q',f(G)) \in F(x_0, x_1, \ldots x_{f(G)+2})$.
  the Lagrange interpolation for $f(G)+1$ points for q, q'
- The final result of the reduction is obtained by evaluating this rational function at

$$x_0 = \chi(G,q'),\ x_1 = \chi(G \bowtie K_1, q'),\ \ \ldots,\ \ x_n = \chi(G \bowtie K_n, q')$$
$$x_{n+1} = q,\ x_{n+2} = q'$$

A suitable model of computation for A is

the unit-cost model BSS
advocated by L. Blum, M. Shub and S. Smale.

The uniform difficult point property for $\chi(G, \lambda)$

(i) We have shown:

For all $q \in \mathbb{Q} - \{0, 1, 2\}$ and $q' \in \mathbb{Q}$ the numeric graph invariants $\chi(-, q)$ and $\chi(-, q')$ polynomial time Turing reducible to each other.

(ii) But we have shown much more:

There is ONE algebraic reduction scheme
for all the instances $\chi(G, q)$ to $\chi(G, q')$,
where q, q' are not in $\mathbb{N}$.

Uniform algebraic reductions for evaluations of graph polynomials.

Let $f = \Phi(G, \bar{q})$ and $g = \Phi(G, \bar{q}')$ two evaluations of the same graph polynomial Φ. We say that f algebraically reduces to g uniformly in $\bar{q}, \bar{q}'$, and we write $f <^{P}_{UA} g$, if there exists

(i) a finite set $\mathcal{A}_\Phi = \{\alpha_1, \ldots, \alpha_a\}$ of size a of polynomial time computable numeric graph invariants $\alpha : Graphs \rightarrow \mathbb{Q}$, depending on Φ only;

(ii) a polynomial time computable family $r_i : i \in \mathbb{N}$ of polynomial time computable graph transductions $r_i : Graphs \rightarrow Graphs$, depending on Φ only;

The family is polynomial time computable in Φ and i.

(iii) a polynomial time computable function $A_\Phi : \mathbb{Q}^a \rightarrow \mathbb{Q}(x_1, x_2, \ldots)$, depending on Φ only;

such that for every $G \in Graphs$ we have that

$$f(G) = A_\Phi(\alpha_1(G), \ldots, \alpha_a(G)) \left(g(r_1(G), \ldots, g(r_{poly(G)}(G), \bar{q}, \bar{q}'\right)$$

The uniform difficult point property UDPP

Let $\Phi(G, \bar{x}^m)$ be a graph polynomial in m variables.

$\Phi(G, \bar{x}^m)$ has the uniform difficult point property (DPP) if the following holds:

There exists an **exception set** C_Φ which is a countable union of semi-algebraic sets of dimension $< m$ in the field $\mathbb{C}$, and for all q not in the exception set C, $\Phi(-q)$ is $\sharp\mathbf{P}$ hard.

Furthermore, for any $\bar{q}_1, \bar{q}_2 \in F^m - C_\Phi$ we have

$$\Phi(G, \bar{q}_1) <^P_{UA} \Phi(G, \bar{q}_2).$$

In other words, all the evaluations for $\bar{q}$ not in the exception set, are of the same difficulty and uniformly algebraically reducible to each other.

The Tutte polynomial

The paradigm of the DPP was inspired by the work of
Linial and Jaeger, Vertigan and Welsh.

(i) For the classical Tutte polynomial, the **uniform DPP** was proven by Jaeger, Vertigan and Welsh in 1990.

(ii) For the colored Tutte polynomial as defined by Bollobás and Riordan (1999), the **uniform DPP** was proven by Bläser, Dell and Makowsky in 2007.

(iii) This also holds for the multivariate Tutte polynomial, the Pott's model, if restricted to a fixed finite number of variables.

More polynomials with the uniform DPP

The uniform DPP was also proven for

(i) the cover polynomial $C(G, x, y)$ introduced by Chung and Graham (1995) by Bläser Dell, 2007

(ii) the interlace polynomial (aka Martin polynomial) introduced by Martin (1977) and independently by Arratia, Bollobás and Sorkin (2000), by Bläser and Hoffmann, 2007

(iii) the matching polynomial, by Averbouch, Kotek and Makowsky, 2007

(iv) the harmonious chromatic polynomial, by Averbouch, Kotek and Makowsky, 2007

What is the pattern behind this?

In establishing the UDPP one uses the fact that in the examples the evaluations at integer points are in $\sharp\mathbf{P}$.

We call such graph polynomials counting.

There seems to be dichtomy property:

- Either all the evaluations of a graph polynomial Φ are polynomial time computable, or
- Φ has the uniform difficult point property UDPP.

Conjecture: This dichtomy holds for all counting $\mathbf{MSOL}$-definable graph polynomials.

Note that it holds for the harmonious chromatic polynomial, which is not $\mathbf{MSOL}$-definable.

Thank you for your attention !