# Topology of definable groups

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- $\mathcal{M}$: an o-minimal expansion of a real closed field.
- Definable means definable in $\mathcal{M}$.
- Let $X \subset M^n$ be a definable set. Let

$$X = \dot{\bigcup}_{C \in \mathcal{D}_X} C$$

(cell decomposition of $X$).
The o-minimal Euler Characteristic of $X$ is

$$E(X) := \sum_{C \in \mathcal{D}_X} (-1)^{dim(C)}.$$

- If $X = |K|(M)$, with $K$ a closed simplicial complex, then

$$E(X) = \chi(K).$$

A definable group is equipped with a definable manifold topology making "$\cdot$" and $-^{-1}$' continuous.

**Examples.**

- $(M, +)$
- $(M^*, \cdot)$
- $([0,1)(M), +(mod\, 1))$
- $M$-rational points of an algebraic group
- A compact real Lie group.

$G$: A definably connected definable group

**WLOG.** The topology of $G$ is induced by that of the ambient space $M^n$.

All definable maps are suppose to be continuous.

# Definable-tori

A definable-torus (of $G$) is a definably connected definably compact Abelian group (subgroup of $G$).
It doesn't need to be a direct product of circle groups by Peterzil-Steinhorn'99.

**Theorem 1** — Berarducci'08, Strzebonski'94

Let $T$ be a definable-torus of a definably compact $G$. Then,

- for every definable proper subgroup $H$ of $T$, $E(T/H) = 0$;
- $T$ is a maximal definable-torus of $G$ iff $E(G/T) \neq 0$.

**Theorem 2** — Berarducci'08, Edmundo'05

Let $G$ be definably compact. Then,

- for each maximal definable-torus $T$ of $G$,
  $$G = \bigcup_{g \in G} gTg^{-1};$$
- all maximal definable-tori of $G$ are conjugate, hence
  $$Z(G) = \bigcap \{T \colon T \text{ is a maximal definable-torus of } G\}.$$

**Corollary 1**

If $G$ is definably compact then $G/Z(G)$ is centreless.

Proof. $G/Z(G)$ is a definably compact definably connected group.
Let $gZ(G) \neq Z(G)$. Then there is a maximal definable-torus $T$ of $G$ with $g \notin T$.
Since $T$ contains $Z(G)$, $T/Z(G)$ is a definable-torus of $G/Z(G)$ and $gZ(G) \notin T/Z(G)$.
Since

$$E\left(G/Z(G)\big/T/Z(G)\right) = E(G/T) \neq 0,$$

$T/Z(G)$ is a maximal definable-torus of $G/Z(G)$, so $gZ(G)$ cannot be in the centre of $G/Z(G)$. □

**Corollary 2**

$G$ definably compact. $T$ a maximal definable-torus of $G$. Then $C(T) = T$.

*Proof.* First note that for any $a \in G$, $a \in C(a)^0$. Indeed, let $T_1$ be any torus of $G$ such that $a \in T_1$. Then, $a \in T_1 \subset C(a)$. Since $T_1$ is connected $T_1 \subset C(a)^0$.
Let $a \in C(T)$ then $T \subset C(a)^0$. $C(a)^0$ is a definably connected definably compact group, hence $T$ is also a maximal definable-torus of $C(a)^0$.
Since $a \in C(a)^0$, there is a $g \in C(a)^0$ such that $gag^{-1} \in T$. Therefore, $a \in T$. □

**Theorem 3** Berarducci-Ot'09

Let $T$ be a definable-torus definably acting on a definable set $X$. Then $E(X) = E(X^T)$, ($X^T$ is the fixed point set of the action).

Proof. For every $x \in X$, $E(T/T_x) = E(orb(x))$.
Hence, if $|orb(x)| > 1$ then $E(orb(x)) = 0$. □

**Corollary 1**

Let $G$ be definably compact. If $T$ is a definable-torus of $G$ then $E(N(T)/T) = E(G/T)$, if $T$ is a maximal definable-torus of $G$ then $|N(T)/T| = E(G/T)$

Proof. Consider the action of $T$ on the definable set $G/T$ by left multiplication.
Then $(G/T)^T = N(T)/T$.
If $T$ is moreover maximal then $N(T)/T$ (the Weyl group of $G$) is finite (Berarducci'08). □

**Corollary 2**

Let $T$ be a definable-torus of a definably compact $G$. Then,
$T$ is a maximal definable-torus of $G$ iff $[C(T) : T]$ is finite.

**Question**

Let $T_1$ and $T_2$ be two definable-tori of the same dimension. Are $T_1$ and $T_2$ definably homeomorphic?

**Theorem 4** Berarducci-Mamino-Ot.'09

Let $T_1$ and $T_2$ be two definable-tori of the same dimension. Then, $T_1$ and $T_2$ have the same o-minimal homotopy type. That is, there are definable maps $f\colon T_1 \to T_2$ and $g\colon T_2 \to T_1$ such that $gf$ and $fg$ are definably homotopic to $id_{T_1}$ and $id_{T_2}$, respectively.

To prove this we need some preliminary results.

# o-minimal vs. semialgebraic homotopy

Let $X(M)$ and $Y(M)$ be semialgebraic sets, definably compact and defined w/out parameters.

**Theorem 5** Baro-Ot.'09

- For every $f\colon X(M) \to Y(M)$ *definable*, there is a *semialgebraic* $g\colon X(M) \to Y(M)$ defined w/out parameters such that $f \overset{def}{\sim} g$.
- Let $f, g\colon X(M) \to Y(M)$ be semialgebraic (defined w/out parameters). If $f \overset{def}{\sim} g$ then there is a semialgebraic homotopy $H : f \overset{s.a.}{\sim} g$, (resp. with $H$ defined w/out parameters).

In particular, there is a natural bijection:

$$\{f : X(M) \to Y(M) |\, f \text{ s.a.}\}/_{\overset{s.a.}{\sim}} \to \{f : X(M) \to Y(M) |\, f \text{ def}\}/_{\overset{def}{\sim}}$$

$$[f] \longrightarrow [f]$$

# Proof of Thm 5

One of the main ingredients of the proof is the following result.

**Normal Triangulation Theorem** Baro'08

Let $K$ be a closed simplicial complex in $\mathcal{M}$ and let $X_1, \ldots, X_r$ be definable subsets of $|K|$. Then, there is a subdivision $K'$ of $K$ and a definable homeomorphism $\phi : |K'| \to |K|$ such that

- $(K', \phi)$ is a triangulation of $|K|$ partitioning $X_1, \ldots, X_r$ and all $\sigma \in K$, and
- $\phi$ is definably homotopic to $id_{|K|}$.

# o-Minimal homotopy groups

**Definition**

Let $Z$ be a definable set and $z_0 \in Z$. The o-minimal $n$-th homotopy group is

$$\pi_n(Z, z_0) := \{f\colon (I^n, \partial I^n) \to (Z, z_0) \mid f \text{ definable } \}/ \overset{def}{\sim},$$

for each $n > 0$, where $I = [0,1] \subset M$:

**Theorem 6** — Berarducci-Mamino-Ot.'09

$\pi_n(G)$ is a finitely generated Abelian group, for all $n > 0$.

We already knew it for $n = 1$ and also that $\pi_n(G)$ is an abelian group for all $n > 0$.

# Proof of $\pi_n(G)$ fin.gen.

**Definition**

A (definable) $H$-space is a pointed (resp. definable) space $(X, x_0)$ equipped with a (resp. definable) continuous map $\mu\colon X \times X \to X$ such that both maps $\mu(-, x_0)$ and $\mu(x_0, -)$ are (resp. definably) homotopic to $id_X$.

Any definable group is a definable $H$-space. A definable $H$-space defined over $\mathbb{R}$ is indeed an $H$-space.
By taking a definable deformation retract of (a triangulation of) $G$, we obtain a closed simplicial complex $K$.
On $|K|(M)$ the multiplication of $G$ has become a definable map

$$m : |K|(M) \times |K|(M) \to |K|(M)$$

which gives to $(|K|(M), e)$ a structure of definable $H$-space.
WLOG $e$ has rational coordinates.

# Proof of $\pi_n(G)$ fin.gen.

By Theorem 5 we have

- $m$ definably homotopic to some *semialgebraic*

$$\mu : |K|(M) \times |K|(M) \to |K|(M)$$

  defined w/out parameters.
- $\mu(-, e)$ and $\mu(e, -)$ are homotopic to $id_X$ via a semialgebraic homotopy defined w/out parameters.

This gives to $(|K|(M), e)$, a structure of a semialgebraic *$H$*-space with all maps defined w/out parameters.

## Proof of $\pi_n(G)$ fin.gen. Transfer to the reals

**Claim**

Let $|K|(\mathbb{R})$ be the realization of $K$ in $\mathbb{R}$. Then $\pi_n(|K|(\mathbb{R}))$ is a fin. gen. abelian group for all $n > 0$.

Proof of claim. The realization $\mu(\mathbb{R})$ of $\mu$ in $\mathbb{R}$ gives to $|K|(\mathbb{R})$ a structure of $H$-space.
$|K|(M)$ definably connected implies $|K|(\mathbb{R})$ path-connected. In a path-connected $H$-space, the fundamental group acts trivially on all the homotopy groups.
Since $K$ is a finite simplicial complex, the homology groups $H_n(|K|(\mathbb{R})), \quad n > 0$ are finitely generated, and by a classical result all this implies that also the homotopy groups $\pi_n(|K|(\mathbb{R}))$ are finitely generated. □

## Proof of $\pi_n(G)$ fin.gen. — Transfer back to $\mathcal{M}$

Once we have $\pi_n(|K|(\mathbb{R}))$ is a fin. gen. abelian group for all $n > 0$, we transfer back the results on $\mathbb{R}$ to our o-minimal structure $\mathcal{M}$.

**Corollary to Theorem 5**

$$\pi_n(|K|(\mathbb{R})) \cong \pi_n(|K|(M)),$$

for each $n > 0$.

(The r.h.s. is the o-minimal homotopy group and the l.h.s is the topological homotopy group.)

Proof of corollary. By theorem 5 and results of Delfs and Knebusch linking the semialgebraic and the topological setting. □

Since $\pi_n(G) \cong \pi_n(|K|(M))$, this ends the proof of thm 6 □

# Corollaries of Thm 6

**Corollary 1**

If $G$ is Abelian then $\pi_n(G) = 0$, for each $n > 1$.

Proof. Since $\pi_n(G)$ is a f.g. Abelian group, it suffices to prove that it is divisible for each $n > 1$.
For each $k > 0$, consider the map

$$p_k \colon G \ni x \mapsto kx \in G,$$

which is a definable covering map and hence the induced maps

$$\begin{array}{rccc} (p_k)_* \colon & \pi_n(G) & \longrightarrow & \pi_n(G) \\ & [\gamma] & \to & k[\gamma] \end{array}$$

are isomorphism for each $n > 1$. □

# Corollaries of Thm 6

Let $\mathbb{T}^d(M) = [0,1)^d(M)$ with addition modulo 1.

**Corollary 2**

If $G$ is a definable-torus then $G$ has the same o-minimal homotopy type than $\mathbb{T}^d(M)$, where $d$ is the dimension of $G$.

Proof. Let $[\gamma_1], \ldots, [\gamma_d]$ freely generate the abelian group $\pi_1(G)$. Then, the map

$$f \; : \mathbb{T}^d(M) \longrightarrow G : (t_1, \ldots, t_n) \mapsto \gamma_1(t_1) + \ldots + \gamma_n(t_d)$$

is a definable homotopy equivalence. Indeed, $f$ induces an isomorphism on the o-minimal fundamental groups and since all the higher o-minimal homotopy groups are trivial $f$ also induces an isomorphism on them.
We apply the o-minimal Whitehead thm (Baro-Ot.'09). □

This also ends the proof of thm 4: . □

# Corollaries of Thm 6

**Corollary 3**

If $G$ is a definable-torus then $H_n(G) \cong \mathbb{Z}^{\binom{d}{n}}$, for each $0 < n \leq d$, where $d$ is the dimension of $G$.

Proof. Let $n > 0$.
By Cor 2, the o-minimal homology groups

$$H_n(G) \cong H_n(\mathbb{T}^d(M)).$$

By transfer of homology

$$H_n(\mathbb{T}^d(M)) \cong H_n(\mathbb{T}^d(\mathbb{R})),$$

where the r.h.s. is the topological homology group. Finally

$$H_n(\mathbb{T}^d(\mathbb{R})) \cong \mathbb{Z}^{\binom{d}{n}}.$$

□

# Definable fibrations

**Definition**

Let $E$ and $B$ be definable sets and $p : E \longrightarrow B$ a definable map.

- The map $p$ is a definable fibre bundle with fibre a definable set $F$ if there is a finite open definable covering of $B = \bigcup_{i=1}^{k} U_i$ and for each $i$, a definable homeomorphism $\varphi_i \colon U_i \times F \longrightarrow p^{-1}(U_i)$ such that $p \circ \varphi_i \colon U_i \times F \longrightarrow U_i$ is the projection onto the first factor.
- The map $p$ is a definable fibration if for every definable set $X$, for every definable homotopy $f : X \times I \longrightarrow B$ and for every definable map $g : X \longrightarrow E$ such that $pg = f(-, 0)$ there is a definable homotopy $h : X \times I \longrightarrow E$ such that $ph = f$ ($h$ is a lifting of $f$) and $h(-, 0) = g(-)$.

**Theorem 7** Berarducci-Ot.-Mamino'09

Every definable fibre bundle is a definable fibration.

Proof. Let $p : E \longrightarrow B$ a definable fibre bundle. We have to avoid the use of path-spaces and compact-open topology. So instead of a "lifting function for $p$", we define: for each definable set $X$ and for each definable $f : X \times I \longrightarrow B$ a lifting function for $f$ (and $p$) and prove that $p$ is a definable fibration iff for every definable set $X$ and for every definable $f : X \times I \longrightarrow B$ there is a lifting function for $f$. □

**Corollary**

If $H$ is a definable subgroup of $G$ then the canonical $p : G \longrightarrow G/H$ is a definable fibration.

Proof. $p$ is a definable fibre bundle. □

# Pillay's conjecture

From now on assume that

- $\mathcal{M}$ is sufficiently saturated.
- $G$ definably compact.

**Theorem "Pillay's conjeture" (PC)**

- There is a canonical type-definable divisible subgroup $G^{00}$ of $G$ s.th. $\mathbb{L}G := G/G^{00}$ with the logic topology is a compact Lie group (Berarducci-Ot.-Peterzil-Pillay'05).
- $dim\, G = dim_{Lie}\, \mathbb{L}G$ (Hrushovski-Peterzil-Pillay'08).

# The functor $\mathbb{L}$

We have a functor

$$\mathbb{L} : G \to \mathbb{L}G$$

from the *category of definably compact groups* and definable homomorphism to the *category of compact real Lie groups* and continuous homomorphisms.

- $\mathbb{L}$ "preserves" dimension and connectedness (by PC).
- $\mathbb{L}$ is an exact functor (Berarducci'07).
- $\mathbb{L}$ "preserves" cohomology: $H^n(G) \cong H^n(\mathbb{L}G)$ (Berarducci'09, Edmundo-Jones-Peatfield'08).
- $\langle G, \cdot \rangle \equiv \langle \mathbb{L}G, \cdot \rangle$ (Hrushovski-Peterzil-Pillay'09).

**Theorem 8** Berarducci-Ot.-Mamino'09

$$\pi_n(G) \cong \pi_n(\mathbb{L}G)$$

for all $n > 0$.
(LHS: o-minimal homotopy; RHS: topological homotopy.)

**Cases of the proof**

- $G$ Abelian.
- $G$ semisimple.
- General case.

# Proof of $\pi_n(G) \cong \pi_n(\mathbb{L}G)$ Abelian case

$G$ Abelian

$n = 1$:

$$\pi_1(G) \cong \mathbb{Z}^{dim\, G}$$

(Edmundo-Ot.'04).
Pillays' conjecture implies that $\mathbb{L}G$ is a torus of Lie dimension $dim\, G$, hence $\pi_1(\mathbb{L}G) \cong \mathbb{Z}^{dim\, G}$.

$n > 1$:

- $\pi_n(G) = 0$ by Cor.1 to Thm. 6, and
- $\pi_n(\mathbb{L}G) = 0$ for $n > 1$, since $\mathbb{L}G$ is a torus (and $\pi_n(S^1) = 0$).

# Proof of $\pi_n(G) \cong \pi_n(\mathbb{L}G)$ Semisimple case

$G$ semisimple

(1) WLOG

$$G = G(M)$$

a semialgebraic group defined w/out parameters (Edmundo-Jones-Peatfield'07, Peterzil-Pillay-Starchenko'02).

(2)

$$\mathbb{L}G = G(\mathbb{R})$$

(Hrushovski-Peterzil-Pillay'08).

(3)

$$\pi_n(G) \cong \pi_n(\mathbb{L}G)$$

by Cor. to Thm 5.

# Proof of thm 8: $\pi_n(G) \cong \pi_n(\mathbb{L}G)$ General case

**The structure theorem** Hrushovski-Peterzil-Pillay'09

(1) $[G:G]$ is definable, definably connected and semisimple.

(2) $G$ is an almost direct product of $[G:G]$ and $Z(G)^0$, *i.e.* the homomorphism

$$p\colon [G:G] \times Z(G)^0 \to G : (g,h) \mapsto gh$$

is a definable covering map.

**Theorem 9** Baro'09

$G$ and $[G:G] \times Z(G)^0$ have the same o-minimal homotopy type.

In general, the above map $p$ cannot be used to establish a definable homotopy equivalence between $[G:G] \times Z(G)^0$ and $G$. This also ends the proof of Thm 8. □