Generalized Topological Spaces

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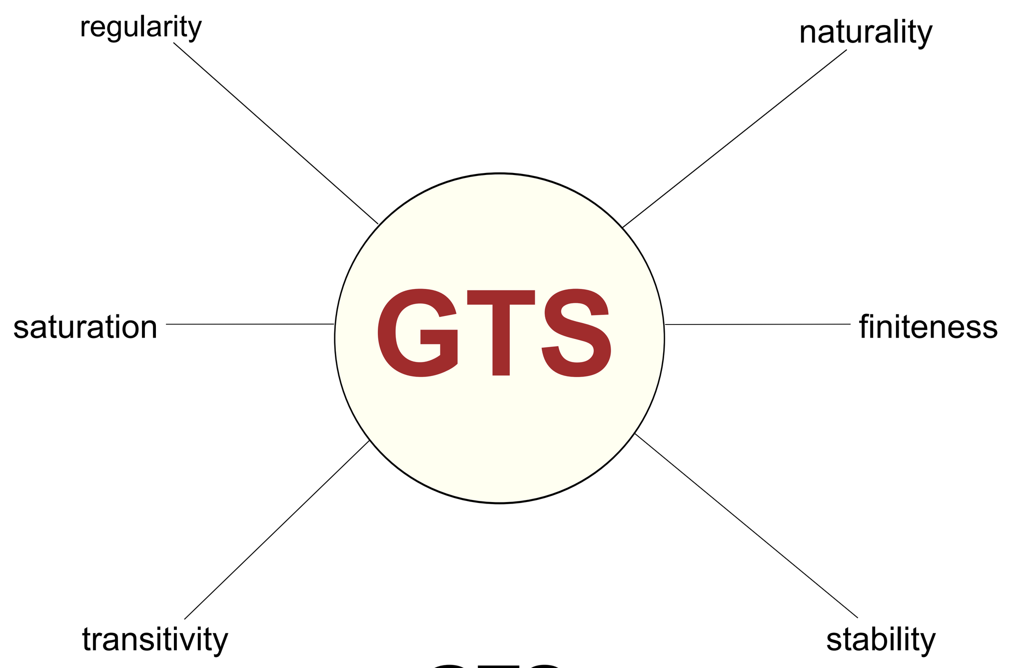

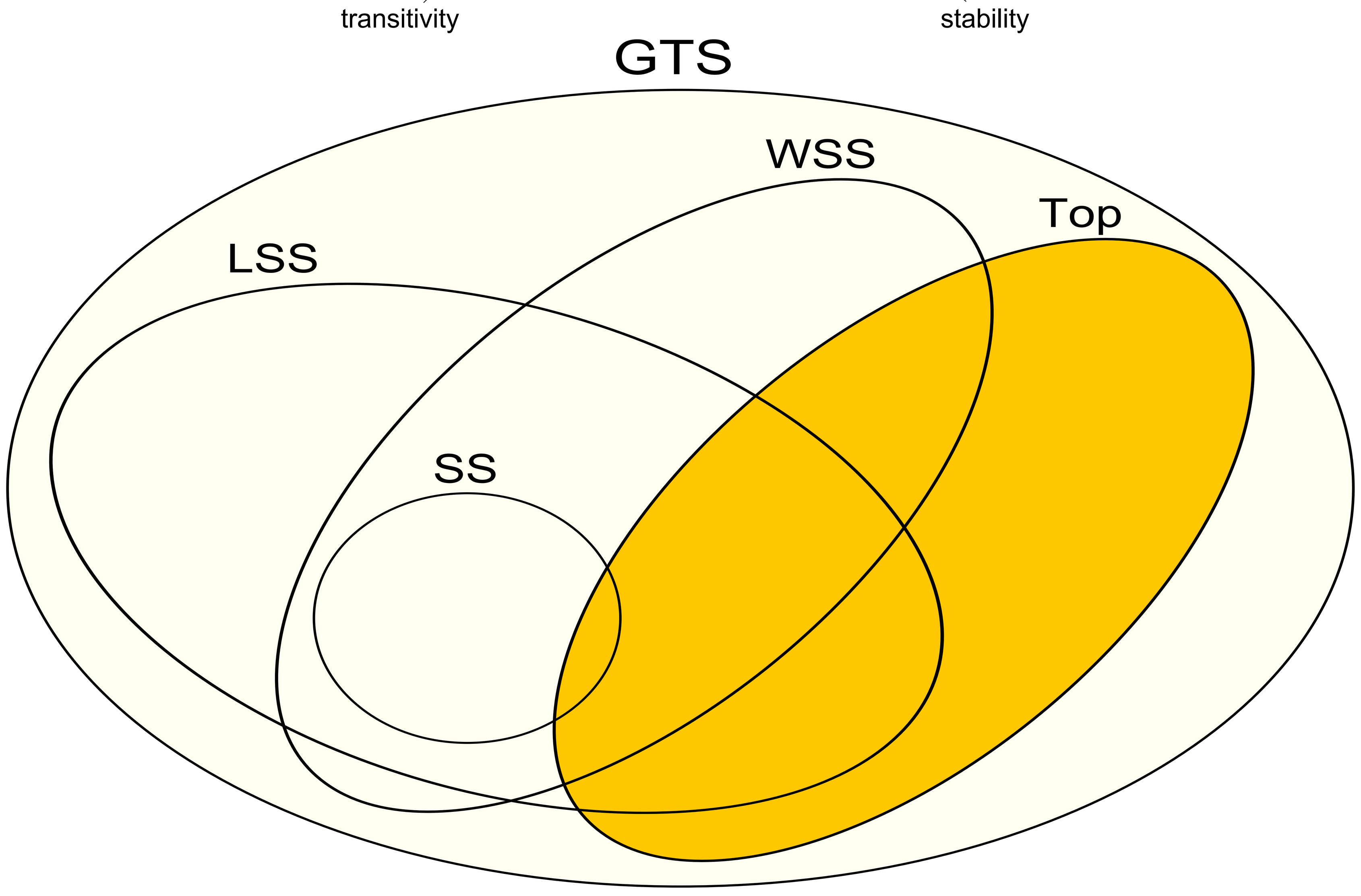

Axioms

A **generalized topological space** is a system

$$(M, \mathrm{Op}(M), \{\mathrm{Cov}_M(U)\}_{U\in\mathrm{Op}(M)})$$

where M is a set, $\mathrm{Op}(M)$ is a family of subsets of M called **open subsets** and $\mathrm{Cov}_M(U)$ for an open U is a family of open families (called **admissible coverings** of U) such that the following axioms are satisfied:

Naturality: each admissible covering is a covering of its union (if $\{U_i\}_{i\in I}\in\mathrm{Cov}_M(U)$, then $\bigcup_{i\in I}U_i=U$);

Finiteness: open sets are stable under finite (including empty) unions and intersections, finite families are admissible coverings of their unions ($\emptyset, M\in\mathrm{Op}(M)$; if $U_1,U_2\in\mathrm{Op}(M)$, then $U_1\cup U_2, U_1\cap U_2\in\mathrm{Op}(M)$; if $\{U_i\}_{i\in I}\subset\mathrm{Op}(M)$ and I is finite, then $\{U_i\}_{i\in I}\in\mathrm{Cov}_M(\bigcup_{i\in I}U_i)$);

Stability: traces of an open set on members of an admissible covering form an admissible covering (if $\{U_i\}_{i\in I}\in\mathrm{Cov}_M(U)$ and $V\in\mathrm{Op}(M)$, then $\{V\cap U_i\}_{i\in I}\in\mathrm{Cov}_M(V\cap U)$);

Transitivity: admissible coverings of members of an admissible covering form together an admissible covering (if $\{U_i\}_{i\in I}\in\mathrm{Cov}_M(U)$ and for each $i\in I$ there is $\{V_{ij}\}_{j\in J_i}\in\mathrm{Cov}_M(U_i)$, then $\{V_{ij}\}_{\substack{i\in I\\ j\in J_i}}\in\mathrm{Cov}_M(U)$);

Saturation: an open family that has a refinement being an admissible covering of their common union is an admissible covering (if $\{U_i\}_{i\in I}\subset\mathrm{Op}(M)$, $U=\bigcup_{i\in I}U_i$, $\{V_j\}_{j\in J}\in\mathrm{Cov}_M(U)$, and $\forall j\in J\,\exists i\in I: V_j\subseteq U_i$, then $\{U_i\}_{i\in I}\in\mathrm{Cov}_M(U)$);

Regularity: a subset of the union of an admissible covering is open if its traces on members of the admissible covering are open (if $\{U_i\}_{i\in I}\in\mathrm{Cov}_M(U)$, $V\subseteq U$ and $V\cap U_i\in\mathrm{Op}(M)$ for each i, then $V\in\mathrm{Op}(M)$).

Morphisms

A **stricly continuous mapping** between *gts*es is such a mapping that the preimage of an open set is an open set and the preimage of an admissible covering is an admissible covering. (They are just *morphisms of sites* in this context.) The *gts*es together with the strictly continuous mappings form a category called **GTS**. The usual topological category **Top** is a full subcategory of **GTS**.

Facts about GTS

1. The category **GTS** has any colimits.
2. The main new notion is that of a small set.
   We call a subset K of a gts M **small** if for each admissible covering of any open U, the set $K\cap U$ has a finite subcovering. (We say that the covering is **essentially finite** on $K\cap U$ or just on K). A subset of a small set is small. The image of a small set by a strictly continuous mapping is small.
3. Importatnt spaces: **locally small** and **weakly small** are "built up" from small spaces by open "admissible coverings" or closed "exhaustions". In a small space "admissible" means "essentially finite". In a locally small space "admissible" means "locally essentially finite". In a weakly small space "admissible" means "piecewise essentially finite".
4. The notions of being **weakly** T_1 (neighborhood separation) and **strongly** T_1 (all singletons closed) need to be distinguished. Similarly with further separation axioms.
5. Products exist in a full subcategory **SS** of small spaces. Finite products exist in full subcategories **LSS** of locally small spaces and $\mathbf{WSS_1}$ of strongly T_1 weakly small spaces.
6. There are different kinds of discrete sets: **weakly discrete** (singletons are open), **discrete** (all sets are open) and **topological discrete** (all families are admissible).
7. Each open subset and each small subset of a *gts* may be treated as a subspace.
8. In a locally small space the **strong topology** is the topology generated by the open sets of the generalized topology. Members of this topology are called **weakly open** subsets. In a weakly small space the **strong topology** is the topology in which the space is the inductive limit of its "pieces" considered with their strong topologies. Each connected component of a locally small or weakly small space is weakly closed.
9. We get functors: the generated (strong) topology functor top : **LSS** $\to$ **Top**, and the strong topology functor $stop$: $\mathbf{WSS_1}\to$ **Top**.

Spaces over structures

Assume that $\mathcal{M}$ is any (first order) model-theoretic structure.

A **function sheaf over** $\mathcal{M}$ on a gts X is a sheaf O_X of sets on X (the sheaf property is assumed only for admissible coverings) such that for each open U the set $O_X(U)$ is contained in the set M^U of all functions from U into M, and the restrictions of the sheaf are the set-theoretical restrictions of functions.

A **space over** $\mathcal{M}$ is a pair (X, O_X), where X is a gts and O_X is a function sheaf over $\mathcal{M}$ on X.

A **morphism** $f:(X,O_X)\to(Y,O_Y)$ of spaces over $\mathcal{M}$ is a strictly continuous mapping $f:X\to Y$ such that for each open subset V of Y the substitution $h\mapsto h\circ f$ gives the mapping $f^{\#}_V: O_Y(V)\to O_X(f^{-1}(V))$. (We could informally say that $f^{\#}: O_Y\to O_X$ is the "morphism of function sheaves" over $\mathcal{M}$ induced by f or define for function sheaves

$$(f_*O_X)(V)=\{h: V\to R|\ h\circ f\in O_X(f^{-1}(V))\}$$

and get the inclusion of function sheaves $f^{\#}: O_Y\to f_*O_X$.)

For each structure $\mathcal{M}$, the category **Space**$(\mathcal{M})$ of spaces over $\mathcal{M}$ and their morphisms has inductive limits.

Facts about Space($\mathcal{M}$)

Assume that a topology on M is given. An **affine definable space** over $\mathcal{M}$ is a space isomorphic to a *gts* as in Fundamental Example with the function sheaf of continuous definable functions. We get also **definable**, **locally definable** and **weakly definable spaces** over $\mathcal{M}$. They form full subcategories $\mathbf{ADS}(\mathcal{M})$, $\mathbf{DS}(\mathcal{M})$, $\mathbf{LDS}(\mathcal{M})$, $\mathbf{WDS}(\mathcal{M})$ of the category **Space**$(\mathcal{M})$. By $\mathbf{WDS}_1(\mathcal{M})$ we denote the full subcategory of (just) T_1 objects of $\mathbf{WDS}(\mathcal{M})$. For each such structure $\mathcal{M}$:

1. finite products exist in $\mathbf{ADS}(\mathcal{M})$, $\mathbf{LDS}(\mathcal{M})$, $\mathbf{WDS}_1(\mathcal{M})$;
2. the categories $\mathbf{LDS}(\mathcal{M})$ and $\mathbf{WDS}_1(\mathcal{M})$ have fiber products;
3. for an object of $\mathbf{LDS}(\mathcal{M})$ or $\mathbf{WDS}_1(\mathcal{M})$, the following conditions are equivalent: being weakly Hausdorff, being strongly Hausdorff, having the diagonal closed.

If $\mathcal{R}$ is an o-minimal expansion of a field, then "definable" over $\mathcal{R}$ versions of arbitrary CW-complexes may be constructed. Then the homotopy categories of topological, semialgebraic, and definable CW-complexes are equivalent.

Examples

1. Fundamental Example. Assume $\mathcal{M}=(M,...)$ is a structure, and a topology is given on M. We consider the product topology on cartesian powers M^n. If $D\subseteq M^n$ is a definable set, then we can set:
   open subset = relatively open, definable subset;
   admissible covering = essentially finite covering.
   Each such D becomes a small *gts*.
2. "The subanalytic site". If M is a real analytic manifold, then we can set:
   open subset = open subanalytic subset;
   admissible covering = covering that is essentially finite on compact subsets.
   We get a locally small *gts*.
3. Some metic spaces. If a metric space (X,d) satisfies the ball property
   (**BP**) each intersection of two open balls is a finite union of open balls,
   then X has a natural generalized topology, where: Y is an open subset of X if the trace of Y on each open ball is a finite union of open balls, and an open covering is admissible if it is essentially finite on each open ball. Then open balls are small sets. The covering of the space by all open balls is admissible (each small set is covered by one open ball), and X is a locally small *gts* satisfying the first axiom of countability.
4. Infinite discrete small spaces. On any infinite set X, there is still a generalized topology making X a discrete small *gts*. It is enough to set: an open subset is any subset, an admissible family is any essentially finite family. (Remember that all small subsets of the topological space $\mathbb{R}$ are finite.)
5. Weakly discrete spaces Let X be an infinite *gts* where the open sets are finite sets or the whole space, and the admissible coverings are exactly the essentially finite open families. Then X is a weakly discrete space, weakly (not strongly) T_1. The generated topology is discrete.
6. Real lines. The topological real line $\mathbb{R}$ may be given four locally semialgebraic space structures: the affine semialgebraic space $\mathbb{R}$, the "localized" space $\mathbb{R}_{loc}=\overset{a}{\bigcup}_{n\in\mathbb{N}}(-n,n)$, and the "half-localized" spaces $\mathbb{R}_{loc,+}=\overset{a}{\bigcup}_{n\in\mathbb{N}}(-\infty,n)$, $\mathbb{R}_{loc,-}=\overset{a}{\bigcup}_{n\in\mathbb{N}}(-n,+\infty)$.

For the details look at

A. Piękosz, *O-minimal homotopy and generalized (co)homology*, arXiv:0808.3866 [math.LO].

A. Piękosz, *On generalized topological spaces*, arXiv:0904.4896 [math.LO].