Unidimensional simple theories

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Simple theories

A complete first-order theory T is *simple* iff there is an independence relation (i.e. an invariant, symmetric and transitive relation with extension, finite character and local character) $a \underset{A}{\downarrow} B$, for a and $A \subseteq B$ from the monster model $\mathcal{C}$ of T, that satisfies the independence theorem over a model.

$a \underset{A}{\downarrow} B$ iff for every formula $\phi(x, y) \in L$ if $\phi(a, B)$ then $\phi(x, B)$ doesn't fork over A.

From now on T denotes a simple theory in a language L, and $\mathcal{C}$ denotes a fixed monster model of T.

Simple theories

Definition

The SU-rank of $tp(a/A)$ is defined by induction on α: if $\alpha = \beta + 1$, $SU(a/A) \geq \alpha$ if there exists $B \supseteq A$ such that $a \underset{A}{\not\downarrow} B$ and $SU(a/B) \geq \beta$; if α is a limit ordinal, $SU(a/A) \geq \alpha$ if $SU(a/A) \geq \beta$ for all $\beta < \alpha$.

T is called *supersimple* if $SU(p) < \infty$ for every $p \in S_x(A)$, for all A and finite x.

Definition

A formula $\phi(x, y) \in L$ is *low in x* if there exists $k < \omega$ such that for every $\emptyset$-indiscernible sequence $(b_i | i < \omega)$, the set $\{\phi(x, b_i) | i < \omega\}$ is inconsistent iff every (some) subset of it of size k is inconsistent. T is *low* if every formula is.

Simple theories

I'll say that a theory is *hypersimple* if it is simple and it eliminates hyperimaginaries, that is, on a complete type every type-definable equivalence relation is an intersection of definable equivalence relations.

Definition

$p \in S(A)$ is said to be *orthogonal* to $q \in S(B)$ if $a \underset{C}{\downarrow} b$ whenever $tp(a/C)$ is a non-forking extension of p and $tp(b/C)$ is a non-forking extension of q.

T is called *unidimensional* if any two non-algebraic types are non-orthogonal.

Unidimensionality and supersimplicity

Problem (Shelah)

Is any unidimensional stable theory supersimple?

Theorem (Hrushovski - 1986)

Yes.

First, a proof in case L is countable [H0] and then in full generality [H1].

Unidimensionality and supersimplicity - the simple case

Theorem (S. - 2003)

A small simple unidimensional theory is supersimple [S3].

Theorem (Pillay - 2003)

A countable hypersimple low unidimensional theory is supersimple.

(proved in [P] with additional assumption that removed later.)

In this talk I'll give an overview of the following new result [S4]:

Theorem (S. - 2008)

A countable hypersimple unidimensional theory is supersimple.

Analyzability - basic notions

From now on assume T is a hypersimple theory and work in $\mathcal{C}^{eq}$. Let $p \in S(A)$ and let $\mathcal{U}$ be an A-invariant set.

We say that p *is (almost-)* $\mathcal{U}$*-internal* if there exists a realization a of p and there exists $B \supseteq A$ with $a \underset{A}{\downarrow} B$ such that $a \in dcl(B\bar{c})$ (respectively, $a \in acl(B\bar{c})$) for some tuple $\bar{c}$ of realizations of $\mathcal{U}$.

We say that p *is (almost-) analyzable in* $\mathcal{U}$ *in* α *steps* if there exists a sequence $I = (a_i | i \leq \alpha) \subseteq dcl(a_\alpha A)$, where $a_\alpha \models p$, such that $tp(a_i / A \cup \{a_j | j < i\})$ is (almost-) $\mathcal{U}$-internal for every $i \leq \alpha$.

Fact

Assume T is a hypersimple unidimensional theory, p is non-algebraic, and $\mathcal{U}$ is unbounded. Then
1) for $a \models p$ there exists $a' \in dcl(Aa) \setminus acl(A)$ such that $tp(a'/A)$ is $\mathcal{U}$-internal.
2) p is analyzable in $\mathcal{U}$.

Analyzability - basic notions

We say that the SU-rank of $\mathcal{U}$ is $\leq \alpha$, and write $SU(\mathcal{U}) \leq \alpha$, if $Sup\{SU(p) | p \in S(A), p^{\mathcal{C}} \subseteq \mathcal{U}\} \leq \alpha$.

We say that $\mathcal{U}$ *is supersimple* if there exists $\alpha \in On$ such that $SU(\mathcal{U}) \leq \alpha$. We say that $\mathcal{U}$ *has bounded finite SU-rank* if $SU(\mathcal{U}) \leq n$ for some $n < \omega$.

Fact

Assume p is almost-analyzable in $\mathcal{U}$ in finitely many steps and $\mathcal{U}$ is supersimple. Then $SU(p) < \infty$.

A basic fact about internality and compactness yield:

Fact

Assume $p \in S(A)$ is analyzable in an A-definable set $\mathcal{U}$. Then p is almost-analyzable in $\mathcal{U}$ in finitely many steps (in fact, p is analyzable in $\mathcal{U}$ in finitely many steps, but this is harder).

The forking topology

Finding directly a non-algebraic supersimple definable set seems inaccessible. To resolve this, in [H0] and then in [P] new topologies have been introduced. Indeed, the main role of these topologies in their proof was the ability to express the relation $\Gamma_S(x)$ defined by

$$\Gamma_S(x) = \exists y(S(x,y) \wedge \ y \downarrow x \ )$$

as a closed relation for any Stone-closed relation $S(x,y)$.

For the general case this topology will not be sufficient for producing a supersimple set directly. However, we will use a variant of this topology for a different purpose first:

The forking topology

Definition

Let $A \subseteq \mathcal{C}$. An A-invariant set $\mathcal{U}$ is said to be *a basic τ^f-open set over A* if there is a $\phi(x, y) \in L(A)$ such that

$$\mathcal{U} = \{a | \phi(a, y) \text{ forks over } A\}.$$

Note that the family of basic τ^f-open sets over A is closed under finite intersections, thus form a basis for a unique topology on $S_x(A)$. Clearly, the τ^f-topology refines the Stone-topology. If $acl_x(A)$ is infinite, the set $\{a \in \mathcal{C}^x | \ a \notin acl(A)\}$ is an example of a τ^f-open set over A that is not Stone-open over A.

Definition

We say that the τ^f-topologies over A are closed under projections (T is PCFT over A) *if for every τ^f-open set $\mathcal{U}(x, y)$ over A the set $\exists y \mathcal{U}(x, y)$ is a τ^f-open set over A. We say that* the τ^f-topologies are closed under projections (T is PCFT) *if they are over every set A.*

Analyzability in the forking topology

The following result reduces the main problem to the problem of producing an unbounded supersimple τ^f-open set, provided that we know PCFT.

Fact (S2)

Assume T is a simple theory with PCFT. Let $p \in S(A)$ and let $\mathcal{U}$ be a τ^f-open set over A. Suppose p is analyzable in $\mathcal{U}$. Then p is analyzable in $\mathcal{U}$ in finitely many steps.

Unidimensionality, EPFO and PCFT

Recall the following notion from [BPV].

Definition

We say that the extension property is first-order in T (I'll say T is EPFO) if for every formulas $\phi(x, y), \psi(y, z) \in L$ the relation $Q_{\phi,\psi}$ defined by:

$$Q_{\phi,\psi}(a) \text{ iff } \phi(x, b) \text{ doesn't fork over } a \text{ for every } b \models \psi(y, a)$$

is type-definable (here a can be an infinite tuple from $\mathcal{C}$ whose sorts are fixed).

Remark

Note that if T is EPFO then T eliminates $\exists^\infty$. Thus by Shelah's fcp theorem we get that a stable theory with EPFO has the nfcp. The converse is also true. This suggests an analogue notion for simple theories. If T is low and EPFO then we say T has the wnfcp [BPV].

Unidimensionality, EPFO and PCFT

Fact (S1)

Let T be a simple unidimensional theory. Then T eliminates $\exists^\infty$.

Corollary (Pillay)

A simple unidimensional theory is EPFO.

Lemma

Let T be a simple theory with EPFO. Then T is PCFT.

Corollary

Let T be a simple unidimenisonal theory. Then T is PCFT.

Corollary

Let T be a hypersimple unidimensional theory. Let $p \in S(A)$ and let $\mathcal{U}$ be an unbounded τ^f-open set over A. Then p is analyzable in $\mathcal{U}$ in finitely many steps. In particular, for such T the existence of an unbounded supersimple τ^f-open set over some set A implies T is supersimple.

Existence of an unbounded supersimple τ^f-open set

At this point it easy to conclude that any countable hypersimple low unidimensional theory is supersimple. Indeed, the existence of such a set follows by Hrushovski's Baire categoricity argument [H0] applied to the τ^f-topology using PCFT:

Fix a non-algebraic sort s. W.l.o.g. there is $p_0 \in S(\emptyset)$ with $SU(p_0) = 1$. For all $\emptyset$-definable functions $f(x), g(y, \bar{z})$ let

$$F_{f,g} = \{a \in \mathcal{C}^s |\ f(a) = g(b, \bar{c}) \notin acl(\emptyset) \text{ for some } \bar{c} \subseteq p_0^{\mathcal{C}} \text{ and some } b \downarrow f(a) \}.$$

By the basic property of the τ^f-topology ($\Gamma_S(x)$ is τ^f-closed whenever $S(x, y)$ is Stone-closed), each $F_{f,g}$ is τ^f-closed. By unidimensionality,

$$\mathcal{C}^s \backslash acl(\emptyset) = \bigcup_{f,g} F_{f,g}.$$

Existence of an unbounded supersimple τ^f-open set

By Baire categoricity theorem, there are f^*, g^* s.t. F_{f^*,g^*} has non-empty τ^f-interior. By PCFT, $f^*[F_{f^*,g^*}]$ contains an unbounded τ^f-open set over $\emptyset$. As each $d \in f^*[F_{f^*,g^*}]$ is internal in p_0, $f^*[F_{f^*,g^*}]$ has finite SU-rank (in fact, bounded finite SU-rank).

The reason this argument works is that the τ^f-topology in a low theory is a Baire space (as basic τ^f-open sets are type-definable).

Existence of an unbounded supersimple τ^f-open set

The general case requires some new technologies. Generally, it is achieved via the dividing line "T is essentially 1-based" which roughly means that every type is analyzable by types that are 1-based up to a nowhere dense error in the sense of the forking topology:

A dichotomy for projection closed topologies

Definition

A family

$$\Upsilon = \{\Upsilon_{x,A} | \ x \text{ is a finite sequence of variables and } A \subset \mathcal{C} \text{ is small}\}$$

is said to be *a projection closed family of topologies* if each $\Upsilon_{x,A}$ is a topology on $S_x(A)$ that refines the Stone-topology on $S_x(A)$, this family is invariant under automorphisms of $\mathcal{C}$ and change of variables by variables of the same sort, the family is closed under product by the Stone spaces $S_y(A)$ (where y is a disjoint tuple of variables), and the family is closed under projections, namely whenever $\mathcal{U}(x,y) \in \Upsilon_{xy,A}$, $\exists y \mathcal{U}(x,y) \in \Upsilon_{x,A}$.

Remark

There are two natural examples of projections-closed families of topologies; the Stone topologies and the τ^f-topologies of a theory with PCFT.

A dichotomy for projection closed topologies

From now on fix a projection closed family of topologies Υ.

Definition

1) A type $p \in S(A)$ is said to be *essentially 1-based by mean of* Υ if for every finite tuple $\bar{c}$ from p and for every type-definable Υ-open set $\mathcal{U}$ over $A\bar{c}$, the set $\{a \in \mathcal{U} | \ a \not\downarrow_{acl^{eq}(Aa) \cap acl^{eq}(A\bar{c})} A\bar{c}\}$ is nowhere dense in the Stone-topology of $\mathcal{U}$.

2) Let V be an A-invariant set and let $p \in S(A)$. We say that p is *analyzable in* V *by essentially 1-based types by mean of* Υ if there exists a sequence $(a_i | \ i \leq \alpha) \subseteq dcl^{eq}(Aa_\alpha)$ with $a_\alpha \models p$ such that $tp(a_i/A \cup \{a_j | j < i\})$ is V-internal and essentially 1-based by mean of Υ for all $i \leq \alpha$.

Example

The unique non-algebraic 1-type over $\emptyset$ in algebraically closed fields is essentially 1-based by mean of the Stone-topologies but not by mean of the τ^f-topologies.

A dichotomy for projection closed topologies

Theorem

Let T be a countable hypersimple theory. Let Υ be a projection-closed family of topologies. Let p_0 be a partial type over $\emptyset$ of SU-rank 1. Then, either there exists an unbounded finite-SU-rank (possibly with no finite bound) Υ-open set, OR every type $p \in S(A)$, with A countable, that is internal in p_0 is essentially 1-based by mean of Υ.
In particular, if T is in addition unidimensional, either there exists an unbounded finite SU-rank Υ-open set, or every $p \in S(\emptyset)$ is analyzable in p_0 by essentially 1-based types by mean of Υ.

Stable SU-rank

Definition

Let $a \in \mathcal{C}$, $A \subseteq B \subseteq \mathcal{C}$. We say that a is *stably-independent from B over A* and write $a \underset{A}{\overset{s}{\downarrow}} B$ if for every stable $\phi(x,y) \in L$, if $\phi(x,b)$ is over B and $a' \models \phi(x,b)$ for some $a' \in dcl(Aa)$, then $\phi(x,b)$ doesn't fork over A.

Definition

1) For $a \in \mathcal{C}$ and $A \subseteq \mathcal{C}$ the SU_{se}-rank is defined by induction on α: if $\alpha = \beta + 1$, $SU_{se}(a/A) \geq \alpha$ if there exist $B_1 \supseteq B_0 \supseteq A$ such that $a \underset{B_0}{\overset{s}{\not\downarrow}} B_1$ and $SU_{se}(a/B_1) \geq \beta$; for limit α, $SU_{se}(a/A) \geq \alpha$ if $SU_{se}(a/A) \geq \beta$ for all $\beta < \alpha$.

2) Let $\mathcal{U}$ be an A-invariant set. We write $SU_{se}(\mathcal{U}) = \alpha$ (the SU_{se}-rank of $\mathcal{U}$ is α) if $Max\{SU_{se}(p) | p \in S(A), p^{\mathcal{C}} \subseteq \mathcal{U}\} = \alpha$. We say that $\mathcal{U}$ *has bounded finite SU_{se}-rank* if for some $n < \omega$, $SU_{se}(\mathcal{U}) = n$.

Stable SU-rank

Remark

Note that for all $a \in \mathcal{C}$ and $A \subseteq B \subseteq \mathcal{C}$:

1) $SU_{se}(a/B) \leq SU_{se}(a/A)$,

2) $SU_{se}(a/A) \leq SU(a/A)$,

3) $SU_{se}(a/A) = 0$ *iff* $a \in acl(A)$.

Lemma

Let T be simple and assume $Lstp = stp$ over sets. Then $\mathop{\smile\!\!\!\!\!\!\;|}^{s}$ *is symmetric.*

Existence of an unbounded τ^f_∞-open set of bounded finite SU_{se}-rank

Definition

The τ^f_∞-topology *on $S(A)$ is the topology whose basis is the family of type-definable τ^f-open sets over A.*

By the dichotomy theorem and the previous corollary it is not hard to conclude:

Lemma

Let T be a countable hypersimple unidimensional theory. Assume there is $p_0 \in S(\emptyset)$ of SU-rank 1 and there exists an unbounded τ^f_∞-open set of bounded finite SU_{se}-rank that is over a finite set. Then T is supersimple.

Existence of an unbounded τ^f_∞-open set of bounded finite SU_{se}-rank

A Baire categoricity argument using an "independence relation" like $\mathop{\smile\!\!\!\!\!\!|}\,^s$ instead of $\mathop{\smile\!\!\!\!\!\!|}$ seemed very natural but doesn't seem to work. The problem is that we need the SU_{se}-rank (or some variant of it) to be preserved in free extensions.

The solution of this obtained by analyzing generalizations of local versions of sets of the form:

$$U_{f,n} = \{a \in \mathcal{C}^s \mid SU_{se}(f(a)) \geq n\}$$

where $n < \omega$ and f is a $\emptyset$-definable function.

$\tilde{\tau}^f$-sets

Definition

A relation $V(x, z_1, ...z_l)$ *is said to be a* pre-$\tilde{\tau}^f$-set relation (of degree l) *if there are* $\theta(\tilde{x}, x, z_1, z_2, ..., z_l) \in L$ *and* $\phi_i(\tilde{x}, y_i) \in L$ *for* $0 \leq i \leq l$ *such that for all* $a, d_1, ..., d_l \in \mathcal{C}$ *we have*

$$V(a, d_1, ..., d_l) \text{ iff } \exists \tilde{a}\, [\theta(\tilde{a}, a, d_1, d_2, ..., d_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}, y_i) \text{ forks over } d_1 d_2 ... d_i)]$$

(for $i = 0$ *the sequence* $d_1 d_2 ... d_i$ *is interpreted as* $\emptyset$*).*

Note that if T is PCFT then V is a pre-$\tilde{\tau}^f$-set relation of degree 0 iff V is a τ^f-open set.

Definition

1) A $\tilde{\tau}^f$-set over $\emptyset$ *is a set of the form*

$$\mathcal{U} = \{a \mid \exists d_1, d_2, ...d_l\ V(a, d_1, ..., d_l)\}$$

for some pre-$\tilde{\tau}^f$-set relation $V(x, z_1, ...z_l)$.

$\tilde{\tau}^f$-sets

Remark

By symmetry of $\downarrow^s$ *,* $U_{f,n}$ *is a union of* $\tilde{\tau}^f$*-sets for all* f, n*.*

The main tool for producing an unbounded τ^f_∞-open set of bounded finite SU_{se}-rank is the following theorem. It says that any minimal unbounded fiber of an unbounded $\tilde{\tau}^f$-set is a τ^f-open set:

Theorem

Assume T *is simple and EPFO. Let* $\mathcal{U}$ *be an unbounded* $\tilde{\tau}^f$*-set over* $\emptyset$*. Then there exists an unbounded* τ^f*-open set* $\mathcal{U}^*$ *over some finite set* A^* *such that* $\mathcal{U}^* \subseteq \mathcal{U}$*. In fact, if* $V(x, z_1, ..., z_l)$ *is a pre-*$\tilde{\tau}^f$*-set relation such that* $\mathcal{U} = \{a | \exists d_1...d_l V(a, d_1, ..., d_l)\}$*, and* $(d^*_1, ..., d^*_m)$ *is any maximal sequence (with respect to extension) such that* $\exists d_{m+1}...d_l V(\mathcal{C}, d^*_1, ..., d^*_m, d_{m+1}, ..., d_l)$ *is unbounded, then*

$$\mathcal{U}^* = \exists d_{m+1}...d_l V(\mathcal{C}, d^*_1, ..., d^*_m, d_{m+1}, ..., d_l)$$

is a τ^f*-open set over* $d^*_1...d^*_m$*.*

$\tilde{\tau}^f$-sets

Building on the previous theorem we show the existence of the required set. This completes the proof of the main result.

Theorem

Let T be a countable simple theory with EPFO. Let s be a sort such that $\mathcal{C}^s$ is not algebraic. Assume for every $a \in \mathcal{C}^s \backslash acl(\emptyset)$ there exists $a' \in dcl(a) \backslash acl(\emptyset)$ such that $SU_{se}(a') < \omega$. Then there exists an unbounded τ^f_∞-open set of bounded finite SU_{se}-rank that is over a finite set.

It is easy to conclude:

Corollary

Let T be a countable theory with nfcp. Let s be a sort such that $\mathcal{C}^s$ is not algebraic. Assume for every $a \in \mathcal{C}^s \backslash acl(\emptyset)$ there exists $a' \in dcl(a) \backslash acl(\emptyset)$ with $SU(a') < \omega$. Then there exists a SU-rank 1 definable set.

Corollary

A countable hypersimple unidimensional theory has the wnfcp.

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