Computable functions on the reals

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Definition (Kontsevich, Zagier)

Periods are values of absolutely convergent integrals over open semialgebraic subsets of $\mathbb{R}^n$ of rational functions with rational coefficients.

Semialgebraic: definable in $(\mathbb{R}, +, \cdot)$
In this talk always assume semialgebraic sets to be definable without parameters.
Open semialgebraic: $\{\bar{x} \colon \bigwedge_{i<k} p_i(\bar{x}) > 0, p_i(\bar{x}) \in \mathbb{Q}[\bar{X}]\}$.

Examples

$\pi, q, \ln(q), q \in \mathbb{Q}$ are periods.

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Question

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Clearly there are only countably many periods and they form a $\mathbb{Q}$-algebra.

Theorem (Yoshinaga)

Periods are elementary real numbers.

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Definition

A class $\mathcal{F}$ of functions $f : \mathbb{N}^n \longrightarrow \mathbb{N}, n \in \mathbb{N}$, is called *good* if it contains

1. the constant functions,
2. the projections $(x_1, \ldots, x_n) \mapsto x_i$,
3. modified difference $x \dot{-} y$.

and is closed under composition and bounded summation, i.e. if $g(\bar{x}, i) \in \mathcal{F}$, then also $f(\bar{x}, k) = \sum_{i=0}^{k} g(\bar{x}, i) \in \mathcal{F}$.

$f : \mathbb{N}^n \longrightarrow \mathbb{N}^m$ is in $\mathcal{F}$ if each $f_i, i = 1 \ldots n$ is.

The smallest good class is the class of *low* functions.

The smallest good class also closed under bounded products

$$f(\bar{x}, y) = \prod_{i=0}^{y} g(\bar{x}, i),$$

or – equivalently – the smallest good class which contains $n \mapsto 2^n$, is the class of *elementary* functions.
The elementary functions are the third class $\mathcal{E}_3$ of the Grzegorczyk hierarchy.
We have:

$$LOW \subsetneq \mathcal{E}_2 \subsetneq \mathcal{E}_3 = ELEM \subseteq \ldots \subsetneq prim.rec. \subsetneq rec$$

f low $\Rightarrow f(n) \leq n^k$ for some k, i.e. polynomially bounded.
f elementary $\Rightarrow f$ hyperexponentially bounded.

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Coding countable sets

Call a set X an $\mathcal{F}$–retract (of $\mathbb{N}^n$) if there are functions $\iota : X \to \mathbb{N}^n$ and $\pi : \mathbb{N}^n \to X$ with $\pi \circ \iota = \mathrm{id}$ and $\iota \circ \pi \in \mathcal{F}$.

We define a function $f : X \to X'$ to be in $\mathcal{F}$ if $\iota' \circ f \circ \pi : \mathbb{N}^n \to \mathbb{N}^{n'}$ is.

$\mathbb{Z}$ is a low retract of $\mathbb{N}$ in a canonical way. We turn $\mathbb{Q}$ into a low retract of $\mathbb{Z} \times \mathbb{N}$ by setting $\iota(r) = (z, n)$, where $\frac{z}{n}$ is the unique representation of r with $n > 0$ and $(z, n) = 1$. Define $\pi(z, n)$ as $\frac{z}{n}$ if $n > 0$ and as 0 otherwise.

Computable reals

Definition

A number $\alpha \in \mathbb{R}$ is $\mathcal{F}$-computable if there is an $\mathcal{F}$–function $f : \mathbb{N} \longrightarrow \mathbb{Q}$ with $|f(k) - \alpha| < \frac{1}{k}$.

(Unfortunately,) many 'natural' numbers are low: π, e etc (Skordev, 2008)

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Computable functions

Notation: For $O \subseteq \mathbb{R}^n$ open and $k \in \mathbb{N}$ define

$$O_k = \{\bar{x} \in O \colon |x| \leq k, dist(\mathbb{R}^n \setminus O, \bar{x}) \leq 1/k\}.$$

Then O_k is compact and

$$O = \bigcup_{k \in \mathbb{N}} O_k.$$

Definition

The function $F : O \longrightarrow \mathbb{R}$ is $\mathcal{F}$-computable if there are $\mathcal{F}$-functions $d : \mathbb{N} \longrightarrow \mathbb{N}$ and $f : \mathbb{Q}^n \times \mathbb{N} \longrightarrow \mathbb{Q}$ such that for all $x \in O_k, a \in \mathbb{Q}^n$ we have

$$|x - a| < \frac{1}{d(k)} \Rightarrow |F(x) - f(a, k)| < \frac{1}{k}. \tag{0.1}$$

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Properties of $\mathcal{F}$-computable functions

1. $\mathcal{F}$-computable functions are continuous.
2. $\mathcal{F}$-computable functions take $\mathcal{F}$-reals to $\mathcal{F}$-reals.
   If $|\alpha - g(k)| < 1/k$ and f, d are witnesses for F to be in $\mathcal{F}$, then $|F(\alpha) - f(g(d(k)), k)| < 1/k$ for sufficiently large k.
3. $\mathcal{F}$-computable functions are (somewhat) closed under composition: if $F, G \in \mathcal{F}$, then $F \circ G \in \mathcal{F}$ if F is *uniformly* in $\mathcal{F}$, i.e. if for $x \in O, |x| \leq k, a \in \mathbb{Q}^n$ we have

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Lemma 1

Continuous (0-definable) semialgebraic functions are low on semialgebraic open sets.

(Sketch of proof) Let O be open semialgebraic, F semialgebraic and continuous on O. On O_{2k}, the smallest real valued function d with

$$|x - a| < \frac{1}{d(k)} \Rightarrow |F(x) - F(a)| < \frac{1}{2k}$$

is 0-definable and hence polynomially bounded. Also F is polynomially bounded. Hence for $a \in O_{2k}$ we define $f(a, k)$ as the unique

$$b \in \{-k^m, -k^m + \frac{1}{2k}, \ldots, k^m - \frac{1}{2k}, k^m\}$$

with $F(a) \in [b, b + \frac{1}{2k})$. So if $x \in O_k$ with $|x - a| < \frac{1}{d(k)}$ we have $a \in O_{2k}$ and

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Theorem 2

Let $G, H : O \longrightarrow \mathbb{R}$ be $\mathcal{F}$-functions with $G < H$ on O. Put

$$O_G^H = \{(\bar{x}, y)\colon G(\bar{x}) < y < H(\bar{x})\}.$$

Let $F : O_G^H \longrightarrow \mathbb{R}$ be in $\mathcal{F}$ and assume that $|F(\bar{x}, y)| < \beta(\bar{x})$ for some $\mathcal{F}$-function β.
Then

$$I(x) = \int_{G(x)}^{H(x)} F(x, y)\,\mathrm{d}y$$

is an $\mathcal{F}$–function $O \rightarrow \mathbb{R}$.

Approximate Riemann integral. However, we cannot sum up series of rational numbers as a low function.

Integration Theorem

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Sums of rational numbers

Use:

Lemma

If $g : \mathbb{N}^{n+1} \to \mathbb{Q}$ is in $\mathcal{F}$, there is an $\mathcal{F}$–function $f : \mathbb{N}^n \times \mathbb{N}_{>0} \to \mathbb{Q}$ such that

$$\left| f(x, y, k) - \sum_{i=0}^{y} g(x, i) \right| < \frac{1}{k}.$$

Periods

Theorem (T.-Z., Yoshinaga)

Periods are low reals.

Lemma (Yoshinaga)

Periods are differences of volumes of bounded open 0-definable semialgebraic sets.

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Inverse Function Theorems

Theorem

Suppose c_0, c_1 *are* $\mathcal{F}$*-reals and* $F : O = (c_0, c_1) \longrightarrow V \subseteq \mathbb{R}$ *is a homeomorphism in* $\mathcal{F}$*. Then* F^{-1} *is in* $\mathcal{F}$ *provided there is an* $\mathcal{F}$*-function* $d' : \mathbb{N} \longrightarrow \mathbb{N}$ *such that for all* $y, y' \in V_k$

$$|y - y'| < \frac{1}{d'(k)} \Rightarrow |F^{-1}(y) - F^{-1}(y')| < 1/k. \tag{0.2}$$

Corollary

$exp(x)$ *is low on* $(-\infty, r)$ *for any* $r \in \mathbb{R}$.

(Sketch of proof) $ln : (0, exp(r)) \longrightarrow (-\infty, r)$ is low by Lemma 1, Theorem 2. As $exp(x)$ is bounded on $(-\infty, r)$, it satisfies (0.2).

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If $\alpha \in \mathbb{R}$ is an $\mathcal{F}$-real so is exp(α).

Corollary

The low reals form a real closed field of infinite transcendence degree.

(Sketch of proof) By Lemma 1, the low reals form a field.
As the real zeros of real polynomials are piecewise continuous in the coefficients, these piecewise functions and hence its values on low numbers are low, so the field is real closed.
If $a_0, \dots a_n$ are $\mathbb{Q}$–linearly independent algebraic numbers, then $\exp(a_0), \dots \exp(a_n)$ are low and algebraically independent by Lindemann-Weierstraß.

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If $\alpha \in \mathbb{R}$ is an $\mathcal{F}$-real so is exp(α).

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The low reals form a real closed field of infinite transcendence degree.

(Sketch of proof) By Lemma 1, the low reals form a field.
As the real zeros of real polynomials are piecewise continuous in the coefficients, these piecewise functions and hence its values on low numbers are low, so the field is real closed.
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Note that *exp* is not low on all of $\mathbb{R}$. However, by a variant of the Inverse Function Theorem, *exp* is elementary on $\mathbb{R}$:

If $O \subseteq \mathbb{R}^n$ is $\mathcal{F}$–approximable and $F : O \longrightarrow V \subseteq \mathbb{R}^m$ is a homeomorphism in $\mathcal{F}$ such that F^{-1} satisfies (0.2) and is $\mathcal{F}$–compact (i.e. for some $\beta \in \mathcal{F}$ we have $F^{-1}(V_k) \subseteq O_{\beta(k)}$), then $F^{-1} \in \mathcal{F}$.

In this way, we can also show that the complex function $\exp(z)$ is low on each half-space $\{z \mid \mathrm{Re}(z) < s\}$ and elementary on $\mathbb{C}$ $(= \mathbb{R}^2)$.

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Holomorphic functions

Proposition

Let $F(z) = \sum_{i=0}^{\infty} a_i z^i$ be a complex power series with radius of convergence ρ and let $0 < b < \rho$ be an $\mathcal{F}$–real. Then F restricted to the open disc $\{z\colon |z| < b\}$ belongs to $\mathcal{F}$ if and only if $(a_i b^i)_{i\in\mathbb{N}}$ is an $\mathcal{F}$–sequence of complex numbers.

Lemma (Speed-Up Lemma)

Suppose $(a_n) \in \mathbb{C}$ is a sequence and that $0 < b < 1$ is an $\mathcal{F}$–real. Then $(a_n b^n)$ is an $\mathcal{F}$–sequence if $(a_n b^{2n})$ is.

Theorem (Analytic Continuation)

Let F be a holomorphic function defined on an open domain $D \subset \mathbb{C}$. If F is in $\mathcal{F}$ on some non–empty open subset of D, it is in $\mathcal{F}$ on every compact subset of D.

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Corollary

Let F be holomorphic on a punctured disk $D_\bullet = \{z \mid 0 < |z| < r\}$. If 0 is a pole of F and F is in $\mathcal{F}$ on some non–empty open subset of $D_\bullet$, then F is $\mathcal{F}$ on every proper punctured subdisc $D'_\bullet = \{z \mid 0 < |z| < r'\}$.

(Sketch of proof) If 0 is a pole of order k, $F(z)z^k$ is holomorphic on $D = \{z \, |z| < r\}$. By the theorem $F(z)z^k$ is in $\mathcal{F}$ on any disc $D' = \{z : |z| < r'\}$, $r' < r$. Since z^{-k} is low on $D'_\bullet$, F is in $\mathcal{F}$ on $D'_\bullet$.

Note that if 0 is an essential singularity of F, then F is not low on $D_\bullet$ as otherwise the absolute value of F on $\{z \mid 0 < |z| < \frac{1}{k}\}$ would be bounded by a polynomial in k.

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Zeta–function

Recall that for $\mathrm{Re}(z) > 1$, the Riemann Zeta–function is given by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

The function $(\frac{1}{x})^y$ is low on $(1, \infty) \times \{z \mid \mathrm{Re}(z) > 0\} \subseteq \mathbb{R} \times \mathbb{C}$.
Thus $\frac{1}{n^z}$, $(n = 1, 2, \ldots)$ is a low sequence of functions defined on $\{z \mid \mathrm{Re}(z) > 0\}$.
With $t = \mathrm{Re}(z)$ we have the estimate

$$\left|\zeta(z) - \sum_{n=1}^{N} \frac{1}{n^z}\right| \leq \int_N^{\infty} \frac{1}{x^t}\,\mathrm{d}x = \frac{1}{(t-1)N^{t-1}}.$$

Then $\zeta(z)$ is low on every $\{z \mid \mathrm{Re}(z) > s\}$, $(s > 1)$.

Corollary

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Gamma function and open questions

Similarly, the Gamma function

$$\Gamma(z) = \int_0^\infty t^{-1+z} \exp(-t)\, \mathrm{d}\, t$$

is low on every set $\{z : |z| < r\} \setminus S$ where $S = \{-n \mid n \in \mathbb{N}\}$ denotes the set of poles of Γ.
Since $n!$ grows too fast, Γ is not low on $\mathbb{C} \setminus S$.

Question

- *Is $\zeta(z)$ elementary on $\mathbb{C} \setminus \{1\}$.*
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