ESF-EMS-ERCOM CONFERENCE: PERSPECTIVES IN DISCRETE MATHEMATICS PROBLEM SESSION

1.

Problem 1 (proposed by Boris Bukh)**.** Let G and H be graphs. The *tensor product* of G and H, which is written as $G \otimes H$, is the graph whose vertices are $V(G) \times V(H)$; (u, v) and (u', v') form a *non*–edge of $G \otimes H$ if either uu' is a non-edge of G or vv' is a non-edge of H. If $\alpha(G)$ is the independence number of G, it is clear that $\alpha(G \otimes H) \geq \alpha(G)\alpha(H)$. Lovász [9] showed that

$$\alpha(C_5^{\otimes n}) \leq \left(\sqrt{5}\right)^n \; .$$

This bound is tight since $\alpha(C_5 \otimes C_5) = 5$. In particular

$$\limsup_{n\to\infty} \frac{\alpha(C_5^{\otimes n})}{\left(\sqrt{5}\right)^n} = 1 \; .$$

Question 1.1. *What is the value of*

$$\ell = \lim_{n\to\infty} \frac{\alpha(C_5^{\otimes 2n+1})}{5^n}.$$

The limit does exist since $\alpha(C_5^{\otimes 2n+1}) \geq \alpha(C_5^2)\alpha(C_5^{2n-1})$. One can check that $\alpha(C_5 \otimes C_5 \otimes C_5) = 10$. This gives that $2 \leq \ell \leq \sqrt{5}$.

2.

Problem 2 (proposed by Jacob Fox)**.** Let $G = (V, E)$ be a graph. The density of a set $S \subseteq V$ (namely, $\frac{e(S)}{|S|^2}$) is denoted by $d(S)$. A set $S \subset V$ is called a ε*-quasirandom set* if and only if for all $U \subset S$ such that $|U| \geq \varepsilon|S|$ the following inequality holds:

$$|d(U) - d(S)| < \varepsilon.$$

This notion is the same as saying that U is ε-regular within itself. The following lemma is due to Conlon and Fox [4, Lemma 5.2],

Lemma 1. *For all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that every graph G contains a ε-quasirandom subset with $|S| \geq \delta|V|$.*

In particular, the following bounds on $\delta(\varepsilon)$ are known,

$$2^{\varepsilon^{-\Omega(1)}} \leq \frac{1}{\delta(\varepsilon)} \leq 2^{2^{\varepsilon^{-O(1)}}} \; .$$

Question 2.1. *Improve either the upper or the lower bound on $\delta(\varepsilon)$.*

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Another related question is whether this result can be extended to the hypergraph framework. With this purpose, one needs to define carefully the notion of ε-quasirandomnes for hypergraphs.

3.

Problem 3 (proposed by László Lovász/ Kamal Jain)**.** Let d be a positive integer. Given an orientation of a graph, a *d-vector flow* in G is a set of $|E(G)|$ vectors $v_e \in \mathbb{R}^d$ such that for any vertex $v \in V(G)$,

$$\sum_{e=\overrightarrow{uv}} v_e = \sum_{e=\overrightarrow{vu}} v_e \ .$$

A d unit vector flow is a d-vector flow such that $||v_e||_2 = 1$ for all $e \in E(G)$.

Conjecture 3.1. *Every 2-edge connected graph has a $d = 3$ unit vector flow.*

This can be strengthen if we assume that the connectivity is larger.

Conjecture 3.2. *Every 4-edge connected graph has a $d = 2$ unit vector flow.*

If true these conjectures are best possible. Indeed, if there is a cut edge, it is not possible to have a flow. Moreover, a complete graph on 4 vertices does not have a $d = 2$ unit vector flow.

The conjectures are also known to be true for planar graphs, using the Four Color Theorem. Regarding general graphs Jain proved ten years ago that any 2-edge connected graph has a $d = 7$ unit vector flow (not published). Moreover, Thomassen proved that any 8-edge connected graph has a $d = 2$ unit vector flow.

One important remark is that it is enough to prove the conjecture for 3-regular graphs.

4.

Problem 4 (proposed by Michal Lason)**.** Consider k-colored d-dimensional continuous *necklace*, that is, a cube in $\mathbb{R}^d$ divided into k Lebesgue measurable sets. We want to split the necklace between two thieves using t axis aligned hyperplane cuts, in order that both thieves get the same measure of each color. If $t \geq k$ it is possible to do it (Goldberg and West [6], Alon and West [2]) while if $t < k$ there are some bad configurations. Moreover, if $t = k$, for any composition $t = t_1 + \cdots + t_d$ ($t_i \geq 0$) we can split the necklace in a fair way with t_i hyperplanes aligned with respect to the i-th coordinate.

One may ask if for any k-coloring of $\mathbb{R}^d$ (partition of $\mathbb{R}$ into k measurable sets) there exists a necklace with a fair 2-splitting using at most t axis aligned hyperplane cuts. If $t \geq k$ the answer is positive, while if $t < k - d - 1$ it is negative.

Conjecture 4.1. *For any $k > 0$, $t > 0$ such that $t = k - d - 1$ and any measurable k-coloring of $\mathbb{R}^d$ exists a necklace with a fair 2-splitting using at most t axis aligned hyperplane cuts.*

5.

Problem 5 (proposed by Balázs Szegedy)**.** Let $d \in \mathbb{N}$ be fixed. A *random d-regular graph* is a graph chosen uniformly at random from the set of d-regular graphs on n vertices.

Question 5.1. *Do random d-regular graphs have the* poorest *structure among large girth d-regular graphs?*

Since d does not depend on n the number of short cycles is small and thus, it is known that G, locally, looks like a tree. This is also the case for d-regular graphs with large girth.

Conjecture 5.1. *For every $\varepsilon > 0$ and $k, r \in \mathbb{N}$, there exists n such that if G_1 is a random d-regular graph of size at least n and G_2 is a d-regular graph of girth at least n, then with probability $1-\varepsilon$, any coloring $f : V(G_1) \longrightarrow [k]$ can be modeled with an ε-precision on $V(G_2)$ such that the local statistics in balls of radius r are ε-similar.*

That is, up to ε, they have the same statistics of colored trees of radius r and so on.

Other questions arise:

Question 5.2. *Can we say the same for d-regular expanders or pseudorandom graphs?*

Question 5.3. *What if we replace them by Ramanujan graphs?*

6.

Problem 6. *(proposed by Peter Allen).* A *tight path* $P_\ell^{(k)}$ is a k-uniform hypergraph with vertex set $\{v_1, \dots, v_\ell\}$ such that $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$ is a hyperedge for each $1 \le i \le \ell - k + 1$. The problem is to determine the Turán number for $P_\ell^{(k)}$, namely the maximum number of edges that an n–vertex hypergraph can contain without having a copy of $P_\ell^{(k)}$.

A construction of Győri, Katona and Lemons [7] is the following. Find

$$\big(1 - o(1)\big)\frac{\binom{n}{k-1}}{\binom{\ell-1}{k-1}}$$

sets of size $\ell - 1$ in $[n]$, no two sharing $k-1$ or more vertices. The hypergraph consisting of k-uniform cliques on these sets does not contain $P_\ell^{(k)}$ and has

$$\big(1 - o(1)\big)\binom{\ell-1}{k}\frac{\binom{n}{k-1}}{\binom{\ell-1}{k-1}} = \big(1 - o(1)\big)\frac{l-k}{n-k+1}\binom{n}{k} \approx \big(1 - o(1)\big)\frac{\ell}{n}\binom{n}{k}$$

edges.

This construction is possible for all fixed ℓ due to Rödl's celebrated solution of the Erdős-Hanani problem. Using this Győri, Katona and Lemons showed that for any constant ℓ,

$$\big(1 + o(1)\big)\frac{\ell-k-2}{k}\binom{n}{k-1} \le \mathrm{ex}\big(n, P_\ell^{(k)}\big) \le (\ell-k-2)\binom{n}{k-1}.$$

It would be interesting to obtain a sharp result for constant ℓ. We conjecture that the lower bound is optimal.

For ℓ growing as a function of n, much less is known. The construction of Győri, Katona and Lemons can exist up to $\ell \approx \sqrt{n}$, but for $k \ge 3$ it cannot exist for larger ℓ. As an example in the positive direction, for $k = 3$, we can take as vertex set the points of $\mathbb{F}_q^2$, and the lines in $\mathbb{F}_q^2$ form the desired collection of sets. The negative direction is easy to check.

For $k = 2$ the well-known Erdős-Gallai Theorem [5] shows that

$$\mathrm{ex}\big(n, P_\ell^{(2)}\big) \lesssim \frac{\ell}{n}\binom{n}{2}.$$

Allen, Böttcher, Cooley and Mycroft have obtained the upper bound for general k

$$\mathrm{ex}\big(n, P_\ell^{(k)}\big) \le \frac{\ell}{n}\binom{n}{k} + o(n^k)$$

which, while matching the form of the lower bound above and of the Erdős-Gallai Theorem, is only nontrivial when ℓ grows as a *linear* function of n. In this range of ℓ we know that the construction above *cannot* exist. The best construction we know is the hypergraph on $[n]$ consisting of all edges with at least one vertex in $\big[\frac{\ell}{k} - 1\big]$: so we have

$$\binom{n}{k} - \binom{n - \frac{\ell}{k} + 1}{k} \le \mathrm{ex}\big(n, P_\ell^{(k)}\big) \le \frac{\ell}{n}\binom{n}{k} + o(n^k)\,.$$

It is possible that the lower bound is optimal for $\ell = \Theta(n)$. The upper bound is not optimal.

Question 6.1. *What is* $\mathrm{ex}\big(n, P_\ell^{(k)}\big)$*?*

7.

Problem 7 (proposed by Bartosz Walczak)**.** We say that $a, b \in \mathbb{Z}^w$ are k-crossing if there exist $i, j \in \{1, \dots, w\}$ such that $a[i] - b[i] \ge k$ and $b[j] - a[j] \ge k$. Denote by $f(k, w)$ to the maximum size of a family in $\mathbb{Z}^w$ such that any two vectors are 1-crossing and no two vectors are k-crossing.

On one hand, we know the upper bound $f(k, w) \le k^w$. For instance, fix the remainders modulo k of the coordinates of each vector. Every pair of vectors with the same reminders, being 1-crossing, are also k-crossing. Hence we can have at most one vector for each w–tuple of remainders and there are k^w of them.

On the other hand, $f(k, w) \ge k^{w-1}$. The following construction is an example of a family of size k^{w-1} with such property. Consider the family $\mathcal{F} = \{(a_1, \dots, a_{w-1}, -\sum a_i \,:\, (a_1, \dots, a_{w-1}) \in \mathbb{Z}_{k-1}^{w-1})\}$. This family is clearly non k-crossing and, since all vectors add to zero, $\mathcal{F}$ is 1-crossing.

It is conjectured by Felsner, Krawczyk and Mirek (see [8]) that, indeed, $f(k, w) = k^{w-1}$ and the construction is extremal. However, such structure is not the unique one that outputs the lower bound. The vectors $(1, 1, 1), (0, 1, 2), (1, 2, 0), (2, 0, 1)$ provide such a different such example with $k = 2$ and $w = 3$.

For $k = 1$, $w = 1$ or $w = 2$, the result is easy to check. The first non trivial case, when $w = 3$ was proven by Lason, Mirek, Streib, Trotter and Walczak [8].

Question 7.1. *What are the values of* $f(k, w)$ *for* $k \ge 2$ *and* $w \ge 4$*?*

8.

Problem 8 (proposed by Ervin Győri)**.** *Balister, Győri and Schelp posed this conjecture in* [3]*:*

Conjecture 8.1. *Let* $v_1, \dots, v_{2^{d-1}} \in \mathbb{F}_{2^d}$*, with* $v_i \ne 0$ *such that* $\sum v_i = 0$*. Then, there exists a pairing of* $\mathbb{F}_{\not\leq} = \{x_1, x_1'\}\{x_2, x_2'\} \dots \{x_{2^{d-1}}, x_{2^{d-1}}'\}$ *such that* $x_i + x_i' = v_i$*.*

For example, if $v_i = (1, \dots, 1)$ *for all* $1 \le i \le 2^{d-1}$*, then we pair each vector with its complement.*

9.

Problem 9 (proposed by Noga Alon)). Let $G_{n,k}$ be the graph whose vertices are all binary vectors of length n, where two are adjacent iff the Hamming distance between them is at least $n-k$, where $1 \leq k \leq \sqrt{n}$.

Question 9.1. *What is the chromatic number $\chi(G_{n,k})$ of $G_{n,k}$*

It is known that $\chi(G_{n,1}) = 4$ for all $n \geq 2$, that $\chi(G_{n,k}) \geq k+2$ (by the fact that this graph contains an appropriate Kneser graph, or (a bit better when $n-k$ is odd), by applying the Borsuk-Ulam Theorem in a way similar to the one used in the proofs that the chromatic number of the Kneser graph is large), and that $\chi(G_{n,k}) \leq O(k^2)$. It can also be shown that if we consider only coverings of the set of vertices by independent sets each of which is contained in a Hamming ball with radius smaller than $(n-k)/2$ then the $O(k^2)$-estimate is tight.

Conjecture 9.1. *For n and k as above $\chi(G_{n,k}) = \Theta(k^2)$.*

A motivation for the problem appears in [1].

References

1. N. Alon, A. Hassidim, E. Lubetzky, U. Stav, and A. Weinstein, *Broadcasting with side information*, Proc. of the 49^{th} IEEE FOCS (2008), 823–832.
2. Noga Alon and Douglas B. West, *The Borsuk-Ulam theorem and bisection of necklaces*, Proc. Amer. Math. Soc. **98** (1986), no. 4, 623–628. MR 861764 (88b:05017)
3. Paul N. Balister, Ervin Györi, and Richard H. Schelp, *Coloring vertices and edges of a graph by nonempty subsets of a set*, Eur. J. Comb. **32** (2011), no. 4, 533–537.
4. David Conlon and Jacob Fox, *Bounds for graph regularity and removal lemmas*, (2011).
5. P. Erdős and T. Gallai, *On maximal paths and circuits of graphs*, Acta Math. Acad. Sci. Hungar **10** (1959), 337–356 (unbound insert). MR 0114772 (22 #5591)
6. Charles H. Goldberg and Douglas B. West, *Bisection of circle colorings*, SIAM J. Algebraic Discrete Methods **6** (1985), no. 1, 93–106. MR 772181 (86c:05010)
7. Ervin Győri, Gyula Y. Katona, and Nathan Lemons, *Hypergraph extensions of the erdös-gallai theorem*, Electronic Notes in Discrete Mathematics **36** (2010), no. 0, 655 – 662, ¡ce:title¿ISCO 2010 - International Symposium on Combinatorial Optimization¡/ce:title¿.
8. M. Lasoń, P. Micek, N. Streib, W. T. Trotter, and B. Walczak, *An extremal problem on crossing vectors*, ArXiv e-prints (2012).
9. László Lovász, *On the Shannon capacity of a graph*, IEEE Trans. Inform. Theory **25** (1979), no. 1, 1–7. MR 514926 (81g:05095)