Scientific report - RDSES Short Visit Grant 1347

Purpose of visit

A class of random walks that prefer to travel along paths that they have already traveled before, with that preference given by a reinforcement function, are called *reinforced random walks*. Many interesting properties of these walks remain unproved in dimensions $d > 1$, such as recurrence/transience and monotonicity (in the reinforcement) of the diffusion constant etc.

The purpose of the visit was to complete and propose new joint work with Akira Sakai in which we study a class of random walks with reinforcement, but very short spatial memory, called *senile reinforced random walks*.

Work carried out during the visit

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^d$ such that $0 \notin \mathcal{S}$ and if $x \in \mathcal{S}$ then $-x \in \mathcal{S}$. Further suppose that $\{y \in \mathbb{Z}^d : |y| = 1\} \subset \mathcal{S}$. We say that x is a neighbour of y and write $x \sim y$ if $x - y \in \mathcal{S}$.

Let $D(x) = |\mathcal{S}|^{-1} I_{\{x \in \mathcal{S}\}}$. Then D can be regarded as the transition kernel for a random walk on $\mathbb{Z}^d$. Let $\sigma^2 = \sum_x |x|^2 D(x)$. Let $f : \mathbb{N} \to [-1, \infty)$ and set $S_0 = 0$ and choose the first step of the walk according to D. For the remaining steps of the walk, suppose at time n the walk has traversed the edge (S_{n-1}, S_n) exactly m_n times in the immediate past. Then the next step of the walk is chosen according to $\mathbb{P}(S_{n+1} = S_{n-1}) = \frac{1+f(m_n)}{|\mathcal{S}|+f(m_n)}$, with the remaining probability being uniformly distributed among the other $|\mathcal{S}| - 1$ neighbours of S_n. The process $\{S_n\}_{n \geq 0}$ is called *senile random walk*. An important quantity for this model is the random variable $\tau = \sup\{n \geq 1 : S_m \in \{S_0, S_1\} \ \forall \ m \leq n\}$ corresponding to the number of consecutive traversals of the first edge traversed before leaving. If the reinforcement is sufficiently strong the senile random walk gets stuck traversing a random edge forever.

During the visit we completed our proofs of the main results of our paper. We obtain an exact formula for the Green's function (a power series with important coefficients) for the walk in terms of simple random walk and τ, and use this function (and other methods) to extract asymptotic behaviour of the senile random walk such as recurrence/transience and (sub)diffusive behaviour. The latter indicates the appropriate scaling regime for the model. We also extended most of the results from just nearest-neighbour models to more general walk kernels as defined by D above. In particular this enabled us to identify the features of the underlying random walk model needed for our analysis to remain valid. We also identified the critical constant for linearly reinforced senile random walk $f(n) = Cn$ for a given $\mathcal{S}$.

In addition we discussed future projects, such as the possibility of proving a full scaling limit for the model when the reinforcement is such that the walk does not get stuck on an edge. We expect that Brownian motion is the scaling limit when the reinforcement is sufficiently weak.

Main results obtained

The first result shows that, depending on the reinforcement function and the dimension d and $\mathcal{S}$, the walk gets stuck on an edge or is recurrent or transient. This is also what one would expect for reinforced random walks.

Proposition 0.1. *Let A_i be the event that the senile random walk traverses exactly i edges infinitely often and let $A_{\mathbb{Z}^d}$ be the event that every edge in the edge set of $\mathbb{Z}^d$ generated by $\mathcal{S}$ is traversed infinitely often. Then $\mathbb{P}_f(A_0) + \mathbb{P}_f(A_1) + \mathbb{P}_f(A_{\mathbb{Z}^d}) = 1$ and each is a 0-1 event. Furthermore, $\mathbb{P}_f(A_1) = 1$ if and only if $\mathbb{P}_f(\tau = \infty) > 0$, and if and only if $(1 + f(l))^{-1}$ is summable.*

We use a time change argument to relate the behaviour of the Green's function at its radius of convergence to recurrence/transience of the process $\{S_n\}_{n\geq 0}$. The explicit formula that we derive for this function is then used to prove the following theorem, in which type(I) refers to every vertex being visited infinitely often (or not), and type (II) refers to the expected number of visits to each vertex being infinite (or not).

Theorem 0.2. *Let $SeRW_f$ denote senile random walk in $\mathbb{Z}^d$, with f satsfying $\mathbb{P}_f(\tau = \infty) = 0$. Excluding the degenerate case $|\mathcal{S}| = 2$, $f(1) = -1$, we have the following:*

(1) $SeRW_f$ is recurrent(I)/transient(I) iff $SeRW_0$ is recurrent(I)/transient(I).

(2) When $\mathbb{E}_f[\tau] < \infty$, $SeRW_f$ is recurrent(II)/transient(II) iff $SeRW_0$ is recurrent(II)/ transient(II).

(3) When $\mathbb{E}_f[\tau] = \infty$, $SeRW_f$ is recurrent(II).

We are also able to identify the critical constant of linear reinforcement for type (II) recurrence/transience in dimensions $d > 2$.

The second derivative of the Fourier transform of the Green's function gives information about the quantity $\mathbb{E}[|S_n|^2]$, and enables us to prove the following theorem.

Theorem 0.3. *Suppose that $\mathbb{E}[\tau^{1+\epsilon}] < \infty$. Then*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[|S_n|^2] = \frac{\mathbb{P}(\tau \text{ odd})}{1 - \frac{2}{|\mathcal{S}|}\mathbb{P}(\tau \text{ odd})} \frac{\sigma^2}{\mathbb{E}[\tau]}. \qquad (0.1)$$

It follows from (0.1) that the diffusion constant $v = \lim_{n\to\infty} n^{-1}\mathbb{E}[|S_n|^2]$ is not always monotone in the reinforcement function f. We also show that the scaling regime is different for nearest-neighbour senile random walk in 1 dimension with $f(n) = n$, for which $\mathbb{E}[\tau] = \infty$ and hypergeometric functions arise.

Projected publications/articles

Paper entitled "Senile reinforced random walks".