ESF - Short Visit Grant - Final Report

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I was a visitor at the research institute Eurandom at the Technical University of Eindhoven 1-15 September 2005. The purpose of the visit was to collaborate with Professor Remco van der Hofstad.
Our project concerns percolation on finite graphs. We continue the investigation of Borgs et al. [1, 2, 3], and study the asymptotics of the largest and second largest components in the Hamming graph $H(2, n)$. This is a graph on n^2 vertices each corresponding to one of the distinct n^2 2-vectors with components in the set $\{1, \ldots, n\}$, where a pair of vertices are adjacent if they differ in exactly one co-ordinate, in other words the Cartesian product of two complete graphs on n vertices.
We have found a way to show that with probability tending to 1, above the critical window (as defined in [1]) the largest component has size of the order n^2, that is of the order of the volume of $H(2, n)$. More precisely, we think we can establish a law of large numbers for the giant component in the supercritical phase, and also upper bounds on the size of the second largest component. These results will yield new information on the nature of the phase transition in $H(2, n)$, more precise than what follows in consequence of the results in [1, 2, 3]. Our techniques combine couplings of the growth of graph components with suitable branching processes, couplings of suitable sums of random variables to establish stochastic domination relations, as well as concentration of measure inequalities. We also make use of some of the results from [1, 2, 3] in our proofs. In this way, we are able to describe the geometric structure of large components and use it to estimate the probability of a component growing to a given size.
We have now started writing a paper on the phase transition in the graph $H(2, n)$.
It is possible that we might be able to generalise our results to the Hamming graphs $H(d, n)$ where either one or both parameters d and q tend to infinity.

Another project that we have started working on involves the application of lace expansions to the analysis of phase transitions. It is known there exists an expansion in powers of $1/n$ to all orders for the critical edge probability p_c on the lattice $\mathbb{Z}^n$ and the n-cube. The first three coefficients are the same for the cube and lattice of the same dimension [4, 5]; however, it is suspected (but not proven) that the next one is different. We hope to show that this indeed is the case by analysing a lace expansion different from the one used in earlier work. We expect to write a paper on this subject.

During the visit I also gave a guest lecture at Eurandom on the subject of k-cores in random graphs.

Remco van der Hofstad and myself will continue our collaboration over the coming months. I am planning to make another visit to Eurandom in January/February 2006.

References

[1] C. Borgs, J.T. Chayes, R. van der Hofstad, G. Slade, and J. Spencer, Random subgraphs of finite graphs: I. The scaling window under the triangle condition, *Rand. Struct. Algor.* **27** (2005) 137–184.

[2] C. Borgs, J.T. Chayes, R. van der Hofstad, G. Slade, and J. Spencer, Random subgraphs of finite graphs: II. The lace expansion and the triangle condition, to appear in *Ann. Probab.*.

[3] C. Borgs, J.T. Chayes, R. van der Hofstad, G. Slade, and J. Spencer, Random subgraphs of finite graphs: III. The phase transition for the n-cube, to appear in *Combinatorica*.

[4] R. van der Hofstad and G. Slade, Asymptotic expansions in n^{-1} for percolation critical values on the n-cube and $\mathbb{Z}^n$, *Rand. Struct. Algor.* **27** (2005) 331–357.

[5] R. van der Hofstad and G. Slade, Expansion in n^{-1} for percolation critical values on the n-cube and $\mathbb{Z}^n$: The first three terms, to appear in *Combin. Probab. Comput.*.