The parabolic Anderson model for stretched exponential potentials

Report

The aim of my visit to Eindhoven was a collaboration with Prof. Remco van der Hofstad (Eindhoven University of Technology).

We were planning to study the parabolic Anderson model, that is, the Cauchy problem for the heat equation with random potential on $\mathbb{Z}^d$,

$$\begin{cases} \frac{\partial u}{\partial t}(t,z) &= \Delta^d u(t,z) + \xi(z)u(t,z), \quad t \in (0,\infty), z \in \mathbb{Z}^d, \\ u(0,z) &= \mathbf{1}_0(z), \qquad\qquad\qquad\qquad z \in \mathbb{Z}^d, \end{cases}$$

where

$$(\Delta^d f)(z) = \sum_{y \sim z} [f(y) - f(z)]$$

is the discrete Laplacian, and to look at the case where the potential $\{\xi(z) \mid z \in \mathbb{Z}^d\}$ is a collection of independent identically distributed random variables with intermediately heavy tails. More precisely, we were going to consider potentials close to the stretched exponential ones (that is, the ones with distribution function $F(x) = 1 - e^{-x^\gamma}$) in the case when $\gamma < 1$ and study the long-time behaviour of

$$L_t = \frac{1}{t}\log U(t),$$

where

$$U(t) = \sum_{z \in \mathbb{Z}^d} u(t,z)$$

is the total mass of the solution.

We were aiming to study the first few terms of the almost sure asymptotics and the weak asymptotics of L_t and we were successful in doing so in both cases.

We have computed the first two terms of the almost sure (upper and lower) asymptotics and the first three terms of the weak asymptotics. It turned out that already the second term of the almost sure asymptotics is random. However, the second and even the third term of the weak asymptotics are deterministic and only the fourth term is random. This means that L_t has quite strong fluctuations with quite small probability.

Thus, the main results of our collaboration are the following. Denote

$$\begin{aligned} a_t &= d^{\frac{1}{\gamma}} (\log t)^{\frac{1}{\gamma}}, \\ b_t &= (\log t)^{\frac{1}{\gamma}-1} \log\log t, \\ c_t &= -\frac{d^{\frac{1}{\gamma}}}{\gamma} (\log t)^{\frac{1}{\gamma}-1} \log\log\log t, \\ d_t &= (\log t)^{\frac{1}{\gamma}-1}. \end{aligned}$$

Further, denote

$$\beta' = d^{\frac{1}{\gamma}}(\gamma^{-2} - \gamma^{-1}) \qquad \text{and} \qquad \beta'' = b' + \gamma^{-1} d^{\frac{1}{\gamma}-1}.$$

Theorem 1 (almost sure asymptotics). *Let $\xi(0)$ have a stretched exponential distribution with distribution function $F(x) = 1 - e^{-x^\gamma}$, $\gamma < 1$. Then*

$$\liminf_{t\to\infty} \frac{L_t - a_t}{b_t} = \beta' \qquad \textit{and} \qquad \limsup_{t\to\infty} \frac{L_t - a_t}{b_t} = \beta''$$

almost surely.

Theorem 2 (weak asymptotics). *Let $\xi(0)$ have a stretched exponential distribution with distribution function $F(x) = 1 - e^{-x^\gamma}$, $\gamma < 1$. Then*

$$\frac{L_t - a_t - \beta' b_t}{c_t} \Rightarrow 1$$

weakly.

Additionally, there are two bounds $N_l(t)$ and $N_u(t)$ on L_t such that $N_l(t) < L_t + O(1) < N_u(t)$. Denote

$$\lambda = \gamma d^{1-\frac{1}{\gamma}}.$$

and

$$\mu_l = 2^d d^{d\left(\frac{1}{\gamma}-1\right)} \quad \text{and} \quad \mu_u = 2^d d^{d\left(\frac{1}{\gamma}-1\right)}(1-\gamma)^{-d}.$$

Theorem 3 (randomness of the forth order term in the weak asymptotics). *We have*

$$\frac{N_l(t) - a_t - \beta' b_t - c_t}{d_t} \Rightarrow X_l, \quad \textit{where} \quad F_{X_l}(x) = e^{-\mu_l e^{-\lambda x}},$$
$$\frac{N_u(t) - a_t - \beta' b_t - c_t}{d_t} \Rightarrow X_u, \quad \textit{where} \quad F_{X_u}(x) = e^{-\mu_u e^{-\lambda x}},$$

which implies, in particular, that the fourth term of the weak asymptotic of L_t is non-deterministic.

Our results describe the case of potentials with intermediately heavy tails. The problem has also been solved for potentials with light tails (for bounded potential in [1], for almost bounded in [3], and for double-exponential in [2]), but the case of heavier-tailed potentials is still open. In the future, we plan to continue our collaborations and study the long-time behaviour of the parabolic Anderson model in this remaining case.

The results obtained during my visit to Eindhoven are part of a publication in progress: *Weak and Almost Sure Limits for the Parabolic Anderson Model in the Single Peak Case* by R. v. d. Hofstad, W. König, P. Mörters, and N. Sidorova.

References

[1] M. Biskup, W. König, Long-time tails in the parabolic Anderson model with bounded potential. Ann. Probab. 29:2, 636–682 (2001)

[2] J. Gärtner, S. A. Molchanov, Parabolic problems for the Anderson model. II. Second-order asymptotics and structure of high peaks. Probab. Theory Relat. Fields 111, 17–55 (1998)

[3] R. van der Hofstad, W. König, P. Mörters, The universality classes in the parabolic Anderson model, submitted.