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I visited EPFL in Lausanne, Switzerland, to work with Prof. Thomas Mountford and his coworkers Samir Belhaouari and Glauco Valle on a project regarding the interfaces of one-dimensional voter models. The project was completed at the end of the visit, and the paper titled *Convergence results and sharp estimates for the voter model interfaces* was submitted for publication in the journal *Probability Theory and Related Fields*.

We now describe the main results of the paper. Let $\eta_t \in \{0,1\}^{\mathbb{Z}}$ denote the configuration at time t of the voter model starting with the Heavyside configuration: $\eta_0(x) = 1$ if $x \leq 0$, $\eta_0(x) = 0$ if $x \geq 1$. At each site $x \in \mathbb{Z}$, an independent Poisson clock rings with rate 1, at which time a neighbor y is randomly chosen according to a random walk transition kernel $q(x,y) := p(y-x)$, and the voter model configuration at site x, $\eta_t(x)$, is replaced by $\eta_t(y)$. Assume that $\sum_{x\in\mathbb{Z}} xp(x) = 0$ and $\sum_{x\in\mathbb{Z}} x^2 p(x) = \sigma^2 < \infty$, then at any time $t \geq 0$, the positions of the leftmost 0, $l_t = \inf\{x : \eta_t(x) = 0\}$, and the rightmost 1, $r_t = \sup\{x : \eta_t(x) = 1\}$, are well defined. The voter model configuration between l_t and r_t is called the voter model interface, which is a hybrid zone of 0's and 1's. Let $D([0,+\infty),\mathbb{R})$ be the space of right continuous functions with left limits from $[0,+\infty)$ to $\mathbb{R}$, endowed with the Skorohod topology. Our first main result is

Theorem 0.1 *For the one-dimensional voter model defined as above*

(i) If $\sum_{x\in\mathbb{Z}} |x|^\gamma p(x) < \infty$ *for some* $\gamma > 3$*, then the path distributions on* $D([0,+\infty),\mathbb{R})$ *of*

$$\left(\frac{r_{tN^2}}{N}\right)_{t\geq 0} \quad \text{and} \quad \left(\frac{l_{tN^2}}{N}\right)_{t\geq 0}$$

converge weakly to a one-dimensional σ*-speed Brownian Motion.*

(ii) For $(\frac{r_{tN^2}}{N})_{t\geq 0}$ *and* $(\frac{l_{tN^2}}{N})_{t\geq 0}$ *to converge to a Brownian motion, it is necessary that*

$$\sum_{x\in\mathbb{Z}} \frac{|x|^3}{\log^\beta(|x| \vee 2)} p(x) < \infty \qquad \text{for all } \beta > 1.$$

This extends an earlier result of Cox and Durrett [2] in which they showed that the finite-dimensional distributions of $(\frac{r_{tN^2}}{N})_{t\geq 0}$ and $(\frac{l_{tN^2}}{N})_{t\geq 0}$ are that of a Brownian motion, and a more recent result of Newman, Ravishankar and Sun [3], in which they established the weak convergence in paths space under the assumption $\sum_x |x|^5 p(x) < \infty$. The proof of Theorem 0.1 (i) relies on the duality between voter model and coalescing random walks, and uses a chain argument to decompose events and to fully exploit the coalescing nature of the random walks. Theorem 0.1 (ii) asserts that third moment is, in a crude sense, critical.

In [2], Cox and Durrett also showed that if $\sum_x |x|^3 p(x) < \infty$, then the voter model interface evolves as a positive recurrent Markov chain. This result has been improved recently by Belhaouari and Mountford [1] to models with $\sum_x x^2 p(x) < \infty$. In particular, it implies the existence of an equilibrium distribution for the voter model interface, which we denote by π. Let $\Gamma(\xi)$ denote the size of the interface (i.e., $r_t - l_t + 1$) for the interface configuration ξ. In [2] and [1], it was also shown that with respect to the equilibrium distribution π, the interface size has infinite mean. Our next main result is an extension of this, and gives more precise tail bounds for the equilibrium interface size.

Theorem 0.2 *For the non-nearest neighbor one-dimensional voter model defined as before,*

(i) If $\sum_x x^2 p(x) < \infty$, then there exists $C_1 > 0$ such that for all $M \in \mathbb{N}$

$$\pi\{\xi : \Gamma(\xi) \geq M\} \geq \frac{C_1}{M}. \tag{0.1}$$

(ii) If $\sum_x |x|\gamma^p(x) < \infty$ for some $\gamma > 3$, then there exists $C_2 > 0$ such that for all $M \in \mathbb{N}$

$$\pi\{\xi : \Gamma(\xi) \geq M\} \leq \frac{C_2}{M}. \tag{0.2}$$

(iii) Let $\alpha = \sup\{\gamma : \sum_x |x|^\gamma p(x) < \infty\}$. If $\alpha \in (2,3)$, then

$$\limsup_{n\to\infty} \frac{\log \pi\{\xi : \Gamma(\xi) \geq n\}}{\log n} \geq 2 - \alpha. \tag{0.3}$$

Furthermore, there exist choices of $p(\cdot) = p_\alpha(\cdot)$ with $\alpha \in (2,3)$ and

$$\pi\{\xi : \Gamma(\xi) \geq n\} \geq \frac{C}{n^{\alpha-2}} \tag{0.4}$$

for some constant $C > 0$.

One of the remaining open problems is that, is $\sum_x |x|^3 p(x) < \infty$ a sufficient condition for the convergence of voter model interface boundaries to a Brownian motion? Another question concerns the asymptotic tail distribution of the voter model interface size as in Theorem (iii). It is not clear that in this case there will generally be a power law lower bound. We will address these questions in a future work.

References

[1] S. Belhaouari, T. Mountford: Tightness of the interface for one-dimensional voter models, *Preprint.*

[2] Cox J. T., Durrett R.: Hybrid zones and voter model interfaces, Bernoulli 1(4), 343-370, 1995.

[3] Newman C. M., Ravishankar K., Rongfeng Sun: Convergence of coalescing nonsimple random walks to the Brownian web, *Electronic Journal of Probability* **10**, Paper 2, 2005.