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I visited Friedrich-Alexander University Erlangen-Nuremberg in Erlangen, Germany, to work with Prof. Andreas Greven and Jan M. Swart on a project on renormalization analysis of hierarchically interacting diffusions taking values in the quadrant. This project is joint work with D. A. Dawson, A. Greven, F. den Hollander and J. M. Swart, which we expect to result in a series of three papers. The first paper titled *The renormalization transformation for two-type branching models: I. Construction and basic properties* ([DGHSS]) is in preparation. First we recall some background and then we will outline the main results in the first paper.

We are interested in a system of interacting diffusions $X_\xi(t) = (X_{\xi,1}(t), X_{\xi,2}(t)) \in [0,\infty)^2$ indexed by the hierarchical lattice Ω_N, which evolve according to the coupled SDEs

$$\begin{aligned} dX_{\xi,1}(t) &= C \sum_{\eta\in\Omega_N} a_N(\eta,\xi)(X_{\eta,1}(t) - X_{\xi,1}(t))\,dt + \sqrt{2g_1(X_\xi(t))}\,dB_{\xi,1}(t), \\ dX_{\xi,2}(t) &= C \sum_{\eta\in\Omega_N} a_N(\eta,\xi)(X_{\eta,2}(t) - X_{\xi,2}(t))\,dt + \sqrt{2g_2(X_\xi(t))}\,dB_{\xi,2}(t), \end{aligned} \tag{1}$$

where $\xi \in \Omega_N$, $C > 0$, $a_N(\eta,\xi)$ is a random walk transition kernel on Ω_N, g_1, g_2 are diffusion functions satisfying some regularity conditions, and $\{(B_{\xi,1}, B_{\xi,2})\}_{\xi\in\Omega_N}$ are independent standard Brownian motions. The hierarchical lattice Ω_N is the set $\{\xi = (\xi_i)_{i\in\mathbb{N}} \in \{0,1,\ldots,N-1\}^\infty : \sum_i \xi_i < \infty\}$ with the metric $d(\xi,\eta) = \min\{i \in \mathbb{N}\cup\{0\}\colon\ \xi_j = \eta_j\ \forall\, j > i\}$.

The system (1) arises as the diffusion limit of a two-type branching population model. Ω_N labels the collection of colonies, $(X_{\xi,1}, X_{\xi,2})$ is the mass of the two types of population at colony ξ. The drift term $a_N(\eta,\xi)(X_{\eta,i}(t) - X_{\xi,i}(t))\,dt$ accounts for migration from colony η to colony ξ which couples the SDEs, and the diffusion term $\sqrt{2g_i(X_\xi(t))}\,dB_{\xi,i}(t)$, $i = 1,2$, accounts for branching of population of type i at colony ξ with branching rate $g_i(x_1,x_2)/x_i$.

If we choose $Ca_N(\xi,\eta) = \sum_{k\geq d(\xi,\eta)} cN^{1-2k}$ for $\xi \neq \eta$ and 0 otherwise, and let $(X_{\xi,1}(0), X_{\xi,2}(0)) = (\theta_1,\theta_2)$ for some fixed $(\theta_1,\theta_2) \in [0,\infty)^2$ for all $\xi \in \Omega_N$, then the system (1) is amenable to a renormalization analysis. In particular, we expect the block averages

$$\Big(\bar{X}^{[k]}_{\xi,1}(tN^k), \bar{X}^{[k]}_{\xi,2}(tN^k)\Big) = \frac{1}{N^k} \sum_{\substack{\eta\in\Omega_N \\ d(\xi,\eta)\leq k}} \Big(X_{\eta,1}(tN^k), X_{\eta,2}(tN^k)\Big) \underset{N\to\infty}{\Longrightarrow} \Big(Z^{[k]}_1(t), Z^{[k]}_2(t)\Big), \quad k \in \mathbb{N}, \tag{2}$$

where $\Longrightarrow$ denotes weak convergence, and $(Z^{[k]}_1(t), Z^{[k]}_2(t))$ satisfy the autonomous SDE

$$\begin{aligned} dZ^{[k]}_1(t) &= c(\theta_1 - Z^{[k]}_1(t))dt + \sqrt{(F^{[k]}_c g)_1(Z^{[k]}(t))}dW^{[k]}_1(t), \quad Z^{[k]}_1(0) = \theta_1, \\ dZ^{[k]}_2(t) &= c(\theta_2 - Z^{[k]}_2(t))dt + \sqrt{(F^{[k]}_c g)_2(Z^{[k]}(t))}dW^{[k]}_2(t), \quad Z^{[k]}_2(0) = \theta_2. \end{aligned} \tag{3}$$

The diffusion functions $((F^{[k]}_c g)_1, (F^{[k]}_c g)_2)$ are the kth iterate of a nonlinear renormalization transform of the functions $g = (g_1, g_2)$. More specifically, if we let $\Gamma^{c;g}_\theta$ denote the unique equilibrium distribution of the autonomous SDE

$$\begin{aligned} dX_1(t) &= c(\theta_1 - X_1(t))dt + \sqrt{g_1(X(t))}dW_1(t), \\ dX_2(t) &= c(\theta_2 - X_2(t))dt + \sqrt{g_2(X(t))}dW_2(t), \end{aligned} \tag{4}$$

for some fixed $(\theta_1, \theta_2) \in [0, \infty)^2$, then

$$\begin{aligned}(F_c g)_1(\theta_1, \theta_2) &= \int_{[0,\infty)^2} g_1(x)\Gamma_\theta^{c,g}(dx), \\ (F_c g)_2(\theta_1, \theta_2) &= \int_{[0,\infty)^2} g_2(x)\Gamma_\theta^{c,g}(dx).\end{aligned} \tag{5}$$

The convergence of the block averages in (2) has only been established for mutually catalytic branching diffusions ([CDG04]), but it is believed to hold for a general class of diffusion functions. The goal of our project is to analyze the nonlinear renormalization transform F, which is of independent interest. We will identify its fixed points and fixed shapes and their associated domains of attractions, which constitute the so-called universality classes and completely characterize the large space-time scale behavior of the system (1). Previous renormalization analysis of hierarchically interacting diffusions can be found in the reference, where X_ξ is respectively a diffusion in $[0, 1]$ ([DG93a, DG93b, BCGH95]), a diffusion in $[0, \infty)$ ([DG96, BDGH97]), an isotropic diffusion in compact subsets of $\mathbb{R}^d$ ([HS98]), a mutually catalytic diffusions in $[0, \infty)^2$ ([CDG04]), and a catalytic Wright-Fisher diffusions in $[0, 1]^2$ ([FS05]).

Below we outline the main results in the forthcoming paper [DGHSS], which puts the renormalization transformation F_c on a firm footing. First we need to define a suitable class of diffusion functions so that (4) has a unique weak solution.

Definition 1 *Let $\mathcal{C}$ be the class of functions*

$$g(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x})) = (x_1 h_1(\vec{x}), x_2 h_2(\vec{x})) \tag{6}$$

satisfying

(i) Either $h_1 > 0$ on $[0, \infty)^2$ and h_1 is Hölder on compact subsets of $[0, \infty)^2$, or $h_1(x_1, x_2) = x_2\gamma_1(x_1, x_2)$ with $\gamma_1 > 0$ on $[0, \infty)^2$ and γ_1 is Hölder on compact subsets of $[0, \infty)^2$.

(ii) Either $h_2 > 0$ on $[0, \infty)^2$ and h_2 is Hölder on compact subsets of $[0, \infty)^2$, or $h_2(x_1, x_2) = x_1\gamma_2(x_1, x_2)$ with $\gamma_2 > 0$ on $[0, \infty)^2$ and γ_2 is Hölder on compact subsets of $[0, \infty)^2$.

Recently, Dawson and Perkins ([DP05]) have shown that for $g \in \mathcal{C}$, the martingale problem associated with the SDE (4) is wellposed, which allows us to establish

Theorem 1 *For all $g \in \mathcal{C}$ and all $\vec{\theta} \in [0, \infty)^2$, (4) has a unique equilibrium distribution $\Gamma_\theta^{c,g}$, which is weakly continuous in $\vec{\theta}$. Moreover, for all $\vec{\theta} \in (0, \infty)^2$,*

$$\mathcal{L}(\vec{X}(t)) \underset{t\to\infty}{\Longrightarrow} \Gamma_\theta^{c,g} \qquad \text{for all } \vec{X}(0) \in [0, \infty)^2 \tag{7}$$

($\mathcal{L}$ denotes law).

Theorem 1 allows us to define the renormalization transformation F_c on $g \in \mathcal{C}$ (as defined in (5)) unambiguously. To guarantee that F_c can be iterated, we need to impose further constraints on the diffusion function g.

Definition 2 *For $a \geq 0$, let $\mathcal{H}_a \subset \mathcal{C}$ be the class of all $g \in \mathcal{C}$ satisfying*

$$g_i(x_1, x_2) \leq C(1 + x_1)(1 + x_2) + ax_i^2, \qquad i = 1, 2, \tag{8}$$

for some $0 < C < \infty$ and for all $(x_1, x_2) \in [0, \infty)^2$.

Then, our next result shows that F_c is well-defined on the class $\mathcal{H}_a$ when $a < c$.

Theorem 2 *For all $a \in [0, c)$, $F_c g$ is finite and continuous for all $g \in \mathcal{H}_a$.*

We believe that F_c is smoothing, i.e., $F_c g$ is C^∞ on $(0,\infty)^2$, but at present we have no way of proving this.

Finally, to be able to iterate F_c infinitely often, we need to restrict g to the class $\mathcal{H}_0$. The next claim is the last serious challenge remaining before setting the stage for renormalization analysis.

Claim 1 *The class $\mathcal{H}_0$ is preserved under F_c, i.e., $F_c g \in \mathcal{H}_0$ for all $g \in \mathcal{H}_0$ and all $c > 0$.*

It can be shown that diffusion functions of the form $g = (g_1, g_2) = (b_1 x_1 + c_1 x_1 x_2, b_2 x_2 + c_2 x_1 x_2)$ are fixed points, i.e. $F_c g = g$, and we believe these are the only fixed points in the class $\mathcal{H}_0$. Diffusion functions of the form $g = (g_1, g_2) = (a_1 x_1^2 + b_1 x_1 + c_1 x_1 x_2, a_2 x_2^2 + b_2 x_2 + c_2 x_1 x_2)$ are fixed shapes, more precisely $F_c g = (\lambda_1 g_1, \lambda_2 g_2)$ with $\lambda_1 = \frac{c}{c-a_1}$ and $\lambda_2 = \frac{c}{c-a_2}$ if $a_1, a_2 < c$. When a_1 or $a_2 > 0$, then F_c can only be iterated a finite number of times.

Claim 2 *The only fixed points of F_c in $\mathcal{H}_0$ are the four parameter family of diffusion functions*

$$g = (g_1, g_2) = (b_1 x_1 + c_1 x_1 x_2, b_2 x_2 + c_2 x_1 x_2), \qquad b_1, c_1, b_2, c_2 \geq 0. \tag{9}$$

There are no other fixed shapes of F_c in $\mathcal{H}_0$.

In the ensuing work, we will attempt to verify Claim 1 and Claim 2 and identify the domains of attractions of the fixed points.

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