# ESF Short Visit Scientific Report (Ref: 825 & 826)

Rongfeng Sun

December 4, 2005

I visited Friedrich-Alexander University Erlangen-Nuremberg in Erlangen, Germany, to work with Prof. Andreas Greven on a project on renormalization analysis of hierarchically interacting diffusions taking values in the quadrant. This project is joint work with D. A. Dawson, A. Greven, F. den Hollander and J. M. Swart. A previous ESF short visit (reference 546) to Erlangen was granted for the same project. We refer to the report of that visit for motivation and background.

Recall that the renormalization transformation $F_c$ on pairs of diffusion functions $g = (g_1, g_2)$ is defined as

$$(F_c g)_i(\vec{\theta}) = \int_{[0,\infty)^2} g_i(\vec{x}) \Gamma^{c,g}_{\vec{\theta}}(d\vec{x}), \qquad i = 1, 2, \tag{1}$$

where $\Gamma^{c,g}_{\vec{\theta}}(d\vec{x})$ is the unique equilibrium distribution of the autonomous SDE

$$dX_i(t) = c(\theta_i - X_i(t))dt + \sqrt{2g_i(\vec{X}(t))}\, dB_i(t), \quad i = 1, 2. \tag{2}$$

Previously, we have formulated a class of diffusion functions $\mathcal{C}$ for which (2) has a unique weak solution and a unique equilibrium distribution $\Gamma^{c,g}_{\vec{\theta}}$ for all $\vec{\theta} \in [0,\infty)^2$. This implies that $F_c$ is well-defined on the class $\mathcal{C}$. We recall here the definition of $\mathcal{C}$.

**Definition 1 [Class $\mathcal{C}$]**
*Let $\mathcal{C}$ be the class of functions*

$$g(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x})) = (x_1 h_1(\vec{x}), x_2 h_2(\vec{x})) \tag{3}$$

*satisfying*

*(i) Either $h_1 > 0$ and is locally Hölder on $[0,\infty)^2$, or $h_1(x_1, x_2) = x_2 \gamma_1(x_1, x_2)$ with $\gamma_1 > 0$ locally Hölder on $[0,\infty)^2$.*

*(ii) Either $h_2 > 0$ and is locally Hölder on $[0,\infty)^2$, or $h_2(x_1, x_2) = x_1 \gamma_2(x_1, x_2)$ with $\gamma_2 > 0$ locally Hölder on $[0,\infty)^2$.*

To be able to iterate $F_c$ indefinitely, we formulated the following subclasses of $\mathcal{C}$:

**Definition 2 [Class $\mathcal{H}_a$ and $\mathcal{H}_{0+}$]**
*(i) For $a \geq 0$, let $\mathcal{H}_a \subset \mathcal{C}$ be the class of all $g \in \mathcal{C}$ satisfying*

$$g_1(x_1, x_2) + g_2(x_1, x_2) \leq C(1 + x_1)(1 + x_2) + a(x_1^2 + x_2^2) \tag{4}$$

*for some $0 < C = C(g) < \infty$ and for all $(x_1, x_2) \in [0,\infty)^2$.*
*(ii) Define*

$$\mathcal{H}_{0+} = \bigcap_{a>0} \mathcal{H}_a. \tag{5}$$

It was previously shown that $F_c$ is well-defined on the class $\mathcal{H}_a$ when $a < c$, and we *believe* that

**Conjecture 1** *The class $\mathcal{H}_{0+}$ is preserved under $F_c$, i.e., $F_c g \in \mathcal{H}_0$ for all $g \in \mathcal{H}_0$ and all $c > 0$.*

The main difficulty in verifying this conjecture is to prove that $F_c$ preserves the local Hölder continuity of $g_1(\vec{x})$ and $g_2(\vec{x})$. This was the main focus of the two short visits. We were unable to resolve this issue during these two visits, but we have formulated a strategy employing coupling techniques and potential theoretic arguments for positive recurrent Markov chains.

In the last report, we conjectured that

**Conjecture 2** *The only fixed points of $F_c$ in $\mathcal{H}_{0+}$ are the four parameter family of diffusion functions*

$$g = (g_1, g_2) = (b_1 x_1 + c_1 x_1 x_2, b_2 x_2 + c_2 x_1 x_2), \qquad b_1, c_1, b_2, c_2 \geq 0, \quad (b_1 + c_1)(b_2 + c_2) > 0. \tag{6}$$

*There are no other fixed shapes of $F_c$ in $\mathcal{H}_0$.*

We have been able to verify this conjecture if we impose more regularity conditions on $\mathcal{H}_{0+}$.

**Definition 3 [Class $\mathcal{H}^r_{0+}$]**
*Let $\mathcal{H}^r_{0+}$ be the set of $g \in \mathcal{H}_{0+}$ satisfying*

$$(i) \quad \inf_{\vec{x} \in [s,\infty)^2} g_i(\vec{x}) > 0 \quad \text{for all } s > 0, \quad i = 1, 2, \tag{7}$$

$$(ii) \quad g_1(\vec{x}) + g_2(\vec{x}) \leq K(1 + x_1)(1 + x_2) \quad \text{for some } 0 < K < \infty, \tag{8}$$

$$(iii) \quad \forall\ \vec{z} \in R_\infty, \ \lim_{\vec{x} \to \vec{z}} \frac{g_i}{h_{\vec{z}}(\vec{x})} = \lambda_{i,\vec{z}} \quad \text{for some } \lambda_{i,\vec{z}} \in [0, \infty), \quad i = 1, 2. \tag{9}$$

Within the class $\mathcal{H}^r_{0+}$ we can show

**Theorem 1**
*The only fixed points of $F_c$ in $\mathcal{H}^r_{0+}$ are diffusion functions of the form in (6). There are no fixed shapes in $\mathcal{H}^r_{0+}$ with scaling constants $(\lambda_1, \lambda_2)$ with either $\lambda_1 > 1$, or $\lambda_2 > 1$, or $\lambda_1, \lambda_2 < 1$.*

We still haven't been able to rule out fixed shapes with scaling constants $\lambda_1 < 1$ and $\lambda_2 = 1$, or vice versa.

The main goals we hope to achieve in the near future are:

(i) Verify Conjecture 1 and establish that $F_c$ preserves the Hölder continuity of $(g_1, g_2)$.

(ii) Rule out fixed shapes in $\mathcal{H}^r_{0+}$ with scaling constants $\lambda_1 < 1$ and $\lambda_2 = 1$, or vice versa.

(iii) Relax some, if not all, of the constraints (7–9) imposed in Theorem 1.

(iv) Identify the domains of attraction for the fixed points in (6).