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Rongfeng Sun

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I visited ÚTIA (Institute of Information Theory and Automatization) in Prague, Czech Republic, to work with Dr. Jan M. Swart on the so-called *Brownian net*. Here we describe the main results in the forthcoming paper [4].

The Brownian web consists of a family of one-dimensional coalescing Brownian motions, starting from every point in space and time $\mathbb{R} \times \mathbb{R}$. This object was first proposed by Arratia [1], and later studied by Tóth and Werner [5], and more recently by Fontes, Isopi, Newmand and Ravishankar [2, 3], where a suitable topology was introduced and the term *Brownian web* was coined. One of the main motivations for introducing the Brownian web is that it arises as the diffusive scaling limit of the paths of a system of one-dimensional coalescing random walks, starting from every point in the space-time lattice $\mathbb{Z} \times \mathbb{Z}$ (or $\mathbb{Z} \times \mathbb{R}$ for the continuous time analogue). The Brownian web is believed to be the universal scaling limit of general one-dimensional coalescing systems. Furthermore it serves as the graphic representation for a continuum analogue of the voter model. The *Brownian net* we propose in this work is motivated by the observation that, it is the diffusive scaling limit of the paths of systems of branching-coalescing random walks (with weak branching), starting from every point in the even space-time sublattice $(\mathbb{Z} \times \mathbb{Z})_{\rm even}$ where the two coordinates have the same parity.

First we introduce the system of discrete branching-coalescing random walks that motivated the Brownian net. We define the branching-coalescing random walks through a graphical representation. Consider the even space-time sublattice $(\mathbb{Z} \times \mathbb{Z})_{\rm even} := \{(x,t) \in \mathbb{Z} \times \mathbb{Z} : x + t \text{ is even}\}$. Fix $\epsilon \in [0,1]$, which will be a branching parameter. Independently for each $(x,t) \in (\mathbb{Z} \times \mathbb{Z})_{\rm even}$, with probability ϵ, we draw two directed edges from (x,t): $(x,t) \to (x-1,t+1)$ and $(x,t) \to (x+1,t+1)$; with probability $1-\epsilon$, we draw a single directed edge from (x,t), where $(x,t) \to (x-1,t+1)$ and $(x,t) \to (x+1,t+1)$ is drawn with probability $(1-\epsilon)/2$ each. We denote the resulting random variable taking values in the space of directed edge configurations by $\aleph^\epsilon$. The graphical configuration $\aleph^\epsilon$ determines the evolution of a system of branching-coalescing random walks (which also provides a coupling for all possible initial configurations), where given an initial set of occupied sites $A \subset (\mathbb{Z} \times \mathbb{Z})_{\rm even}$, the set of occupied sites at time t is simply the set of sites in $\mathbb{Z} \times \{t\}$ which lie on a path of directed edges in $\aleph^\epsilon$ originating from some site in A. It is easily seen that $\aleph^\epsilon$ defines a system of branching-coalescing random walks, where independently, at each time each walk with probability $(1-\epsilon)/2$ jumps one unit to the left, with probability $(1-\epsilon)/2$ jumps one unit to the right, and with probability ϵ branches into two random walks with one walk jumping one unit to the left and the other one unit to the right; and whenever two random walks land on the same site, they coalesce instantly. The set of branching-coalescing random walk paths with initial configuration $A \subset (\mathbb{Z}\times\mathbb{Z})_{\text{even}}$ is simply the set of paths of directed edges in $\aleph^\epsilon$ originating from A.

Given $\aleph^\epsilon$, we denote the set of paths of directed edges in $\aleph^\epsilon$ starting from any site in $(\mathbb{Z}\times\mathbb{Z})_{\text{even}}$ by U^ϵ, i.e., the paths of the system of branching-coalescing random walks starting from every site in $(\mathbb{Z}\times\mathbb{Z})_{\text{even}}$. Let S_ϵ denote the space-time scaling $(x,t)\to(\epsilon x,\epsilon^2 t)$, and when applied to a set of paths K, let $S_\epsilon K$ denote the set of paths obtained by scaling the graph of each path in K by the map S_ϵ. Also recall the topology introduced by Fontes, Isopi, Newman and Ravishankar [2, 3] for the Brownian web, where (Π, d) denotes the space of continuous paths starting at some space-time point in $\overline{\mathbb{R}\times\mathbb{R}}$ (a compactification of $\mathbb{R}\times\mathbb{R}$), d is a suitable metric on Π which makes it complete and separable. Then the Brownian web takes values in $(\mathcal{H}, d_{\mathcal{H}})$, the space of compact subsets of (Π, d), where $d_{\mathcal{H}}$ is the Hausdorff metric on compact sets of paths induced by d. $(\mathcal{H}, d_{\mathcal{H}})$ again is a complete separable metric space.

Our main result is the following:

Theorem: As $(\mathcal{H}, d_{\mathcal{H}})$-valued random variables, $S_\epsilon\overline{U^\epsilon}$ (as $\epsilon\to 0$) converges weakly to a $(\mathcal{H}, d_{\mathcal{H}})$-valued random variable $\mathcal{N}$, which we call the *Brownian net*, whose distribution is uniquely determined by the following properties:

(i) For each deterministic $z\in\mathbb{R}\times\mathbb{R}$, almost surely there is a unique leftmost path l_z and a unique rightmost path r_z in $\mathcal{N}$ starting from z, i.e., any other path in $\mathcal{N}$ starting from z is bounded between l_z and r_z.

(ii) For any finite deterministic set of space-time points $\{z_1,\cdots,z_m\}\cup\{\tilde z_1,\cdots,\tilde z_n\}\subset\mathbb{R}\times\mathbb{R}$, $\{l_{z_1},\cdots,l_{z_m}\}\cup\{r_{\tilde z_1},\cdots,r_{\tilde z_n}\}$ is distributed such that each l_{z_i} (resp. $r_{\tilde z_j}$) is distributed as a standard Brownian motion with drift -1 (resp. $+1$) starting from z_i (resp. $\tilde z_j$), paths in $\{l_{z_1},\cdots,l_{z_m}\}\cup\{r_{\tilde z_1},\cdots,r_{\tilde z_n}\}$ evolve independently when they are apart, paths in $\{l_{z_1},\cdots,l_{z_m}\}$ (resp. $\{r_{\tilde z_1},\cdots,r_{\tilde z_n}\}$) coalesce when they meet, and paths in $\{l_{z_1},\cdots,l_{z_m}\}$ and $\{r_{\tilde z_1},\cdots,r_{\tilde z_m}\}$ interact by sticky reflection when they meet.

(iii) For any two deterministic countable dense subsets $\mathcal{D}_{\rm l},\mathcal{D}_{\rm r}\subset\mathbb{R}\times\mathbb{R}$, and for any deterministic countable dense subset $\mathcal{T}\subset\mathbb{R}$, almost surely $\mathcal{N}$ is the closure in (Π,d) of the set of continuous paths obtained by hopping a finite number of times between paths in $\{l_z\}_{z\in\mathcal{D}_{\rm l}}$ and $\{r_z\}_{z\in\mathcal{D}_{\rm r}}$ at times in $\mathcal{T}$.

In addition, more properties for the Brownian net are established, including computing exactly the expected particle density of the random point configuration on $\mathbb{R}$ at time t induced by paths in $\mathcal{N}$ starting before or at time 0. Another interesting object related to the Brownian net is the subset of the Brownian net paths which start at $t = -\infty$, which we call the *backbone* of the Brownian net. The backbone is invariant with respect to space-time translation and time reflection. The point configuration at any fixed time t induced by the paths in the backbone is a Poisson point configuration with intensity 2. The point configuration at time t induced by the paths in the Brownian net starting from a fixed finite space-time region converges weakly to the Poisson point configuration with intensity 2 (in the space of closed subsets of $\mathbb{R}$ with Hausdorff metric) as $t \to \infty$. For more details, see [4].

References

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