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I visited EURANDOM in Eindhoven, the Netherlands, to work with Dr. Rongfeng Sun and Prof. F. den Hollander on renormalization of two-type branching models. This is joint work with Prof. D.A. Dawson and Prof. A. Greven. We aim to write a series of three papers, called *The renormalization transformation of two-type branching models*, parts I–III. Parts I and II have the subtitles: Part I *Construction and basic properties* and Part II *Fixed points and fixed shapes.* Part III will be on convergence and domains of attraction.

In this work, following the earlier work of [BCGdH95, BCGdH97, dHS98, Sch98, CDG04] we study maps where a function, describing the local diffusion matrix of a diffusion process, is mapped into the average of that function with respect to the unique invariant measure of the diffusion process itself. Such mappings arise in the analysis of infinite systems of diffusion processes indexed by the hierarchical group, with a linear attractive interaction between the components [DG93a, DG96, DGV95]. In this context, the mappings are called renormalization transformations. For definiteness, we now give a precise definition of such renormalization transformations in a rather general set-up, and then specialize to the specific two-type branching models we are interested in.

Let $D \subset \mathbb{R}^d$ be nonempty, convex, and open. Let $\mathcal{W}$ be a collection of continuous functions w from the closure $\overline{D}$ into the space M^d_+ of symmetric non-negative definite $d \times d$ real matrices, such that $\lambda w \in \mathcal{W}$ for every $\lambda > 0$, $w \in \mathcal{W}$. We call $\mathcal{W}$ a *prerenormalization class* on $\overline{D}$ if the following three conditions are satisfied:

(i) For each constant $c > 0$, $w \in \mathcal{W}$, and $x \in \overline{D}$, the martingale problem for the operator $A^{c,w}_x$ is well-posed, where

$$A^{c,w}_x f(y) := \sum_{i=1}^{d} c\,(x_i - y_i)\frac{\partial}{\partial y_i} f(y) + \sum_{i,j=1}^{d} w_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j} f(y) \qquad (y \in \overline{D}), \tag{1}$$

and the domain of $A^{c,w}_x$ is the space of real functions on $\overline{D}$ that can be extended to a twice continuously differentiable function on $\mathbb{R}^d$ with compact support.

(ii) For each $c > 0$, $w \in \mathcal{W}$, and $x \in \overline{D}$, the martingale problem for $A^{c,w}_x$ has a unique stationary solution with invariant law denoted by $\nu^{c,w}_x$.

(iii) For each $c > 0$, $w \in \mathcal{W}$, $x \in \overline{D}$, and $i, j = 1, \ldots, d$, one has $\displaystyle\int_{\overline{D}} \nu^{c,w}_x(\mathrm{d}y)|w_{ij}(y)| < \infty$.

If $\mathcal{W}$ is a prerenormalization class, then we define for each $c > 0$ and $w \in \mathcal{W}$ a matrix-valued function $F_c w$ on $\overline{D}$ by

$$F_c w(x) := \int_{\overline{D}} \nu^{c,w}_x(dy) w(y) \qquad (x \in \overline{D}). \tag{2}$$

We say that $\mathcal{W}$ is a *renormalization class* on $\overline{D}$ if in addition:

(iv) For each $c > 0$ and $w \in \mathcal{W}$, the function $F_c w$ is an element of $\mathcal{W}$.

If $\mathcal{W}$ is a renormalization class and $c > 0$, then the map $F_c : \mathcal{W} \to \mathcal{W}$ defined by (2) is called the *renormalization transformation* on $\mathcal{W}$ with *migration constant* c. In (1), w is called the *diffusion matrix* and x the *attraction point.* A diffusion matrix $w \in \mathcal{W}$ such that $F_c w = \lambda w$ for some $c, \lambda \in (0, \infty)$ is called a *fixed shape.* If $\lambda = 1$, it is called a *fixed point.*

Our aim is to construct a renormalization class on $\overline{D} = [0, \infty)^2$ whose elements are diffusion matrices of the form

$$w(x_1, x_2) = \begin{pmatrix} h_1(x_1, x_2)x_1 & 0 \\ 0 & h_2(x_1, x_2)x_2 \end{pmatrix}, \tag{3}$$

where h_1, h_2 are real functions on $[0, \infty)^2$. Diffusions with a diffusion matrix of this for may be interpreted as a pair of two continuous-state branching processes, with the branching rate h_i of the i-th component depending on its own state and on the state of the other component. The cases we want to include are

$$\begin{aligned} h_1(x_1, x_2) &:= a_1 x_1 + b_1 x_2 + c_1, \\ h_2(x_1, x_2) &:= a_2 x_2 + b_2 x_1 + c_2. \end{aligned} \tag{4}$$

If $a_1 = 0 = a_2$, these are fixed points under F_c for each $c > 0$; if $a_1, a_2 \neq 0$ they are fixed shapes. We conjecture that within a large class of models of the form (3), these are the only fixed shapes and fixed points, which are moreover attractive in the sense that iterates $F^{(n)} w := F_{c_{n-1}} \circ \cdots \circ F_{c_0} w$ of arbitrary two-type branching diffusion matrices converge to these special matrices, provided $\sum_n \frac{1}{c_n} = \infty$.

So far, we have managed to identify a suitable large class of diffusion matrices of the form (3), for which we can prove that it is a prerenormalization class, as defined above. This means that fixed points and fixed shapes are well-defined. We have made a lot of progress in proving that the matrices (4) are the only fixed points and fixed shapes in our class. Although there are some technical assumptions one would like to remove, this appears to be essentially solved now.

Unfortunately, we still have not managed to prove that our class is a renormalization class, i.e., that iterates are well-defined. This problem appears to be very tough; it has not been solved for several other classes that have been studied in the past. During my stay at EURANDOM, we developed some ideas for tackling this problem in our set-up. At this moment, it is too early to say whether we will be successfull. We will try to work out our ideas in the coming monhts. We also investigated which of our proofs so far extend to higher dimensions, i.e., to general n-type branching models. We found that the picture for $n \geq 3$ appears to be quite different, which made us decide to stay with two-type models for the present project. This looks like an interesting subject for further work, however.

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