Research Networking Programmes

Short Visit Grant ☐ or Exchange Visit Grant ☒

(please tick the relevant box)

Scientific Report

The scientific report (WORD or PDF file – maximum of eight A4 pages) should be submitted online within one month of the event. It will be published on the ESF website.

***Proposal Title**:* Topological issues arising from lattice polytopes and combinatorial spaces.

Application Reference N°: 4896

1) Purpose of the visit

The goal of the visit was to pursue my investigations related to the enumeration and structure of lattice polytopes with few points in three and more dimensions, trying to combine the area in which I am more familiar with (discrete geometry) to applications in algebraic geometry and techniques from algebraic topology: toric varieties associated to lattice polytopes, study of their topology and other topological invariants from combinatorial data, etc.

I took the occasion to follow several courses and attend different seminars in the host university and in other mathematics research institutions in Berlin (BMS, TU, etc) including a seminar in Topological Combinatorics organized by Pavle Blagojevic. During my stay I attended an advanced course on "Combinatorial and Algorithmic Aspects of Convexity" in Paris and the workshop on "Geometric and Topological Combinatorics" in Oberwolfach.

2) Description of the work carried out during the visit

I stayed at Free University during the Winter Semester, which runs from the beginning of October to mid-February. As I mentioned in the application form, the first three months (October-December) I was funded by the Spanish government. The funding I obtained from ACAT allowed me to prolong this visit until the end of the Semester.

During the stay, I was a visitor at the Discrete Geometry Workgroup, and I had the opportunity to interact with a long list of mathematicians, some of them already mentioned in the application: Professors Günter M. Ziegler, Pavle Blagojević and Christian Haase; Junior Professor Raman Sanyal; PostDoctoral Fellows Hao Chen, Arnau Padrol and Ivan Izmestiev , and the PhD students Moritz Schmitt, Katharina Jochemko, Francesco Grande and Jan Hofmann, among others.

Specially intense and potentially fruitful was the collaboration with Professor Christian Haase and his student Jan Hofmann. Although Haase officially belongs to another working group, he mantains a close academic relationship with the Discrete Geometry group. His research is centered in lattice polytopes and their relation with algebraic geometry. Below I describe in more detail my collaboration with him.

ACTIVITIES DURING THE STAY

Some of the following activities cover the whole Winter Semester, Oct. 2014 – Feb. 2015.

Master Courses:

- Discrete Geometry III. This course was given by Professors Günter M. Ziegler, Matthias Beck and Christian Haase. The topics covered were: Lattice Polytopes and Rational Cones, Ehrhart Theory (Ehrhart polynomial and series, and its geometric interpretation), Combinatorial Reciprocity, Poset Cones, Half-open Decompositions of Poset Cones via Permutation Descents.

- Discrete Geometry I. This course was given by Junior Professor Raman Sanyal (TA Arnau Padrol). The topics covered were: Basic structures in discrete geometry (polyhedra and polyhedral complexes; configurations of points, hyperplanes, subspaces; subdivisions and triangulations (Delaunay, Voronoi)) and Polytope theory (representations and the theorem of Minkowski-Weyl; polarity, simple/simplicial polytopes, shellability; face lattices, f-vectors, Euler- and Dehn-Sommerville; graphs, diameters, Hirsch (ex-)conjecture).

Attendance (and participation) in seminars:

- Seminar “Some Interesting Polytopes”, led by Günter M. Ziegler. Each week, one of the participants of the seminar gives a talk presenting a special polytope (or family of polytopes), its history, properties, importance and applications. A discussion takes place immediately after, both about the mathematical content and the expository aspects of the talk. I presented “the associahedron as a fiber polytope”.

- Discrete Geometry Group Seminar. Each Thursday, the members of the Discrete Geometry Workgroup get together and some of them, or a guest of the group, give a talk about their work. This encourages a dinamic flow of ideas within specific areas, providing quite often new points of view, new approaches that help the research job.
I presented my research on lattice 3-polytopes with few lattice points.

- Top Com (Ober)Seminar. Within the Discrete Geometry Workgroup, there is a small subgroup that is more focused on topological aspects. Every Wednesday there was a Topological Combinatorics seminar in the very same way as the general seminar on Thursday, only this one was addresed to a smaller portion of the group. I have attended several of these talks, in particular that of Isaac Mabillard, Ph. D. student of Uli Wagner, who has been a visitor in the group for part of the months of January and February.

Regular attendance to talks common to all the three universities of Berlin: Free (FU), Humboldt (HU) and Technical (TU):

- BMS Fridays: seminars organized by the Berlin Mathematical School (common PhD program). They take place every other Friday, at Urania Hall in Berlin. The speakers are usually internationally renowned mathematicians, and the topics they present cover all fields of mathematics. Professors, post-docs and PhD students from all over Berlin gather at these talks.

- MDS Mondays: seminars organized by the group Methods for Discrete Structures, common program focused in the field of Discrete Mathematics. Seminars alternate location between the three universities (in particular I have attended all the ones that took place at FU and TU). They consist of two one hour talks, the first one given by a Professor, and the second by a PhD student.

- BMS Days 2015 (16-17 Feb.). The BMS organizes, during the first week after the end of the Winter Semester, two days of talks and mathematical events for prospective students. A large part of students and some Professors also attend these talks.

- BMS Student Conference 2015 (18-20 Feb.). For the rest of this week, the Student Representatives in BMS organize a conference where the speakers, except for the invited speakers, are students.

Attendance to conferences and workshops:

- Participation in the Winter School “Combinatorial and algorithmic aspects of convexity”, that took place at Institut Henri Poincaré in Paris, from the 19th to the 23rd of January, 2015. The speakers were Imre Bárány and Santosh Vempala.

- Oberwolfach Workshop “Geometric and Topological Combinatorics”. During the Workshop I had the opportunity to discuss research with other great mathematicians I was meeting for the first time, like Benjamin Nill and Jesus De Loera.
I also had the priviledge of listening the first official talk about the Counterexample for the Topological Tverberg Conjecture, due to Florian Frick. The talk, of title “Tverberg strikes back”, was the last of a series of four given by Ziegler, Blagojević, Wagner and Frick, respectively. The first two introduced corrected proofs for several related old results; Wagner introduced the work that him and his student Isaac Mabillard developed recently, and that was instrumental in finding the counterexamples.

3) Description of the main results obtained

My paper Lattice 3-polytopes with six lattice points was uploaded to ArXiv in January of 2015, once I was already in Berlin, although the paper was essentially finished before my arrival.
During this stay, and after I presented my Master's Thesis in the Discrete Geometry Seminar, a couple of possible approaches as of how to extend this classification to polytopes with more lattice points have come up. Some of these will be incorporated in the final version of my other paper Lattice 3-polytopes with five lattice points. Both papers, currently under review, will have an acknowledgement to the ESF funding.
One of the tools used in these papers was the volume vector, an almost invariant property of lattice 3-polytopes under unimodular equivalence. In conversations with Arnau Padrol during my stay in Berlin we agreed it might be helpful to study the list of the vectors arising in configurations for n=5 and 6 lattice points, to derive properties of their geometric loci as lattice points, and use it to classify polytopes with more lattice points.

Other lines of investigation that have started during the stay, although closely related to the previous topic, have come up during conversations with Christian Haase and his PhD student Jan Hofmann:
In my Master's Thesis, the starting point was to prove that, for each each $n \geq 4$, there exist finitely many lattice 3-polytopes of size n and width larger than one. In joint work with Ziegler, Haase proved that in dimension 4, there exist infinitely many empty simplices of widths 1 and 2. With computer search, they also found all empty simplicies with normalized volume up to 1000. In this list, the largest simplex has volume 179 and all of them have width 3, except one of width 4. This led them to conjecture that the number of empty 4-simplices of width at least 3 is finite, and that the list they give is complete. Finiteness has been proven by Barile, Bernardi, Borisov and Kantor. While researching this topics, we found some interesting questions that might, eventually, help prove that the list is complete.

The first step is to find the minimal constant A (known finite) such that for each $n \geq 5$, there exist finitely many lattice 4-polytopes of size n and width larger than A. The proof for a specific conjectured value is highly related with hollow (without interior points) 3-polytopes.
Hollow 3-polytopes are not yet classified. However, through personal communication, we have known quite recently that Averkov (in joint work with Krümpelmann and Weltge) has recently proven a very important, and helpful, result that had been open for a while.
It is our intention to use this result to, via some computation, enumerate the complete list of lattice hollow 3-polytopes. Once that is finished, we will be able to find the exact value of the constant A and, maybe, apply this result for some other open questions.

4) Future collaboration with host institution (if applicable)

The collaboration with Haase and Hofmann is ongoing. During my stay there we were able to find a lot of interesting questions that we intend to keep working on via email, Skype, or short visits. In particular, I will be paying short visits to Berlin in late

March and early July, since I will be attending a conference and a summer course there, and we will take the occasion to meet and discuss further.
A short summary of the tasks we want to attack is:

- Prove that a special tetrahedron is the largest 3-polytope with two interior points.
- Develop an efficient computer program that checks whether two polytopes are unimodularly equivalent.
- Classify all lattice hollow 3-polytopes.
- Find the constant A.
- Prove whether Haase's list of empty 4-simplices of width larger than 2 is complete or not.

5) Projected publications / articles resulting or to result from the grant *(ESF must be acknowledged in publications resulting from the grantee's work in relation with the grant)*

As mentioned above, the following two papers will have acknowledgements to the ESF since, although the initial versions were uploaded to the arXiv before the beginning of my stay, both benefited from the discussions there and will have changes derived from these discussions in their final versions:

- M. Blanco and F. Santos, Lattice 3-polytopes with 5 lattice points (arXiv:1409.6701)

- M. Blanco and F. Santos, Lattice 3-polytopes with 6 lattice points (arxiv:1501.01055)

A third paper will presumably result from the ongoing collaboration with Christian Haase and Jan Hofmann.

6) Other comments (if any)