Short Visit Grant ☐ or Exchange Visit Grant ☒

(please tick the relevant box)

Scientific Report

The scientific report (WORD or PDF file – maximum of eight A4 pages) should be submitted online within one month of the event. It will be published on the ESF website.

Proposal Title: Branched covers of contact manifolds

Application Reference N°: 4616

1) Purpose of the visit

The main purpose of the visit was twofold: I had started a project with Paolo Lisca about branched covers of contact 3-manifolds, and a project Bruno Martelli on a combinatorial representation of smooth 4-manifolds.

Paolo and I want to get a better understanding of the relationship between the classical invariants of transverse knots in the standard contact 3-sphere and tightness of the associated branched covers (especially cyclic ones).

What is known so far is that there are two kind of conditions that imply tightness (in fact, strong fillability) of branched covers, and these are quasipositivity and the condition that the self-linking number be equal to twice the Seifert genus, minus 1.

We want to weaken this second condition by replacing the Seifert genus with the slice genus (a condition that is automatically ensured by quasipositivity).

Bruno and I gave a combinatorial construction of 4-manifolds that has its roots in algebraic and tropical geometry. It's been recently shown that every smooth curve in the projective space decomposes as a union of elementary blocks of two types. One of the types is a 4-manifold with codimension-2 corners, that's obtained from the projective plane by removing four generic complex lines. We are studying what kind of smooth 4-manifolds one can obtain by relaxing the (very rigid) gluing condition that algebraic geometry imposes on the blocks. This has the advantage of being relatively easy to describe combinatorially. The driving questions are quite basic: what topological 4-manifolds do we obtain? What kind of smooth structures can we construct? Can we realise arbitrary fundamental group/Betti numbers/signature/intersection form?

2) Description of the work carried out during the visit

1. Branched covers: Paolo and I developed some ideas, which originated from Baldwin's work on the subject. We proved a result (described below) which is of interest on its own, and we believe that it implies tightness for the cyclic covers of the 3-sphere, branched over knots with self-linking number equal to twice the slice genus minus 1. Unfortunately, the available technology doesn't allow us to prove this latter part at the moment (even though experts believe that such results are going to be established).

2. 4-manifolds: Bruno and I had (and still have) many questions to investigate. First, we wanted to get a better grasp on our construction so that we could address them better. We played a lot with the combinatorics and, at the same time, managed to understand the building block itself better. We managed to prove that we can realise all fundamental groups with our construction. We also have a concrete description of the block itself which displays its symmetries and properties.

3. Symplectic fillings: Paolo and I started a new project, that is the classification of symplectic fillings of some contact 3-manifolds that Paolo has studied in one of his recent papers. We're currently working on the construction of symplectic caps for these manifolds, and we're planning to use these caps to set in motion the usual machinery for the classification of fillings, applying classical theorems of McDuff and Donaldson.

4. Genus bounds: During the visit, I met one of Lisca's student, Daniele Celoria, and we started working on bounds of the slice genus coming from Heegaard Floer homology. We gave a couple of seminars on recent results in the area, and independently re-discovered some older results which we later found in the literature.

3) Description of the main results obtained

I will follow here the same order of topics as in the description of the work carried out.

1. Branched covers: The main result we have is the following (sl denotes the self-linking number and g* the slice genus).

Theorem. If K is a transverse knot in the standard contact 3-sphere such that sl(K) = 2g*(K)-1, then there is an exact symplectic cobordism from the connected sum of 2g* copies of S^1xS^2 (with its only tight contact structure) to the cyclic cover Y of S^3 branched over K, such that the concave end is a weak contact boundary.

This would be enough to conclude the proof if the concave end was a strong contact boundary, by capping the concave boundary with a strong filling. Nevertheless, it is believed that a cobordism as the one above induces maps in some Floer-type theory (either Heegaard Floer, or Seiberg-Witten, or embedded contact homology) that preserve the nonvanishing of the contact invariant, and this would immediately prove that Y is tight.

2. 4-manifolds: It's well known that every finitely presented group is realised as the fundamental group of a closed, oriented 4-manifold. We were able to prove that this holds also for the manifolds we obtain with our construction.

Theorem. Given a finitely presented group G, we can construct a closed 4-manifold X (using our combintorial description) such that the fundamental group of X is G.

3. Symplectic fillings: We have started this project only quite recently. We have some partial results pointing in the direction of the following conjecture:

Conjecture. A (sufficiently nice) plumbing of symplectic spheres with $b^2_+ = 1$, has a family of neighbourhoods with *concave* boundary.

We have done some computations and we have proved the conjecture in some special cases (but probably already enough for the classification problem we had in mind).

4. Genus bounds: We worked out many computations, and independently re-discovered some results of Rasmussen, concerning the relationship between correction terms of surgeries along a given knot and its slice genus. We also independently found the following result, which has recently appeared in a preprint by Hedden and Watson.

Theorem (Hedden and Watson, '14). The Alexander polynomial of an L-space knot has nonzero second coefficient.

Moreover, we toyed around with many examples, and were able to find a knot with an unexpected property:

Example. There is a knot K such that both $d_1(K)$ and $d_1(m(K))$ are nonvanishing.

Here d_1 is the concordance invariant defined by Peters, that is the correction term of the 3-manifold obtained by doing +1-surgery along K. To the best of our knowledge, no such example appeared in the literature or in any preprint.
This story has strong ties with singularities of complex curves in the plane, and we've just started to study these aspects, in collaboration with some people at my home institution (Rényi Institute).

4) Future collaboration with host institution (if applicable)

I am definitely planning to continue the collaboration on all of the four projects.

1. Branched covers: As said above, this project has hit a "technical wall". We decided to pause and move on to some other questions, but we will keep the problem in mind. We already wrote down the results and constructions we have, and they're ready to be completed as soon as the technology reaches the right point.

2. 4-manifolds: We are planning to write down what we have so far, and we have many more questions to investigate. The next steps to take are in the following two directions: we will look for an algorithm to produce a handle decomposition of a 4-manifold, given one of our combinatorial representation; we will try to understand how the combinatorics determines the intersection form, especially in the simply-connected case.

3. Symplectic fillings: The first issue at hand is the proof of the conjecture stated above, and the precise definition of what "sufficiently nice" means and how weak this can be made. Once we have this, we will apply it to the classification problem for fillings of the manifolds we were originally interested in; to do this, we need to understand the linear algebra of the intersection forms involved.
The next step is to look for more applications of the conjecture, which we expect to be quite general and that could, in principle,

have interesting corollaries (not only with respect to fillings, but also to closed symplectic 4-manifolds).

4. Genus bounds: We are currently trying to figure out whether there are applications of the results we re-discovered to the topology of singular curves in the complex plane, following ideas developed by Borodzik and Livingston in a recent paper.

5) **Projected publications / articles resulting or to result from the grant *(ESF must be acknowledged in publications resulting from the grantee's work in relation with the grant)***

Bruno Martelli and I are planning to start writing our material up soon (but we don't have a title yet).
Paolo Lisca and I have something written down already for the problem on branched double covers, but this is not going to appear as a preprint anytime soon; the work on symplectic fillings looks very promising, and we are planning to have something written down as soon as we have more concrete results to show for.
The project with Daniele Celoria is currently at a stage where the actual outcome can't be easily predicted: we obtained interesting results which have either been in print for a long time or just appeared; the new directions are still at an early stage, and it is unclear what we will be able to prove and whether it will be of interest to the community.

6) **Other comments (if any)**

This has been for me a great opportunity to work on these topics with some continuity, and I'd like to thank you for supporting me.