Project Report

Vera Vértesi

Project title:	Links in lattice homology
Place of visit:	Alfréd Rényi Institute of Mathematics, Budapest Low dimensional Topology (LDT) research group
Host:	András Stipsicz
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The aim of my visit was to "*define multi-filtrations on lattice homology given by a link*" and to "*prove the equivalence of Heegaard Floer homolgy to lattice homology for all negative definite plumbing trees*". During my visit to Budapest we managed to do the first point, and a portion of the second one:

- we managed to generaralise the definition of knot lattice homology for links, and defined *link lattice homology*;
- understood a surgery formula for link lattice homology.

The above two points already give a formula for computing lattice homology starting from a simple piece and then applying the surgery formula. Unfortunatelly the defined link lattice homology does not "look like" the usual version of link Floer homology. Remember that link Floer homology for an l-component link L is an $\mathbb{F}_2[U_1, \dots, U_l]$-vectorspace with l many filtartions (one for each link component). On the other hand link lattice homology is an $\mathbb{F}_2[U]$-vectorspace with 2^l many filtrations (one for each sublink of L). During my visit

- we defined a good candidate in Heegaard Floer homology that is an $\mathbb{F}_2[U]$-vectorspace with 2^l many filtrations (one for each sublink of L).

So what is left to do is:

- understand that the above candidate admits a similar surgery formula to the one in lattice homology

Starting from negative definiet plumbing tree with at most one "bad vertex" and performing surgeries on the corresponding three manifold the above four statements would imply that *Heegaard Floer homology is equivalent to lattice Floer homology for any graph manifold.* And thus would give a very simple algorithm to compute Heegaard Floer homology for graph manifolds. Although the last statement feels hard to tackle (compare with the recent result of C. Manolescu and P. Ozsváth) we can show that it is enough to prove the statement in a simple case where the link has only two components consisting of an arbitrary knot and its meridian.

In the following I will give a short overview of lattice homology and the set of filtrations defined on it. Given a tree G with $vert(G) = V \cup W$ and fixed weights $n: V \to \mathbb{Z}$ such that the induced subgraph $\Gamma = (G|_V, n)$ is negative definite. The tree Γ defines a four manifold

X_Γ with boundary Y_Γ. The extra vertices of W define a $|W|$-component link $L \subset Y_\Gamma$. Lattice homology of the 3-manifold Y_Γ is the homology of the chain complex $\mathbb{CF}^-(\Gamma)$ generated over $\mathbb{F}_2[U]$ by the pairs $[K, E]$, where K is a characteristic cohomology class in $H^2(X_\Gamma; \mathbb{Z})$, E is a subset of V (by an abuse of notation it also denotes the element $E = \sum_{v \in E} v \in H_2(X_\Gamma; \mathbb{Z})$). The differential of an element is given by the formula:

$$\partial[K, E] = \sum_{v \in E} U^{a_v[K,E]}[K, E - v] + U^{b_v[K,E]}[K + 2v^*, E - v]$$

where v^* is the characteristic element corresponding to the Poinntcaré dual of v in X_Γ (i.e. $v^*(u) = v \cdot u$) and

$$\begin{aligned} 2a_v[K, E] &= \min_{I \subset E - v}\{K(I) + I^2\} - \min_{I \subset E}\{K(I) + I^2\} \\ 2b_v[K, E] &= \min_{I \subset E - v}\{K(I + v) + (I + v)^2\} - \min_{I \subset E}\{K(I) + I^2\}. \end{aligned}$$

For $X \subset W$ let Σ_X be the unique element of the form $\Sigma_X = X + \sum_{v \in V} c_v v \in H_2(X_\Gamma; \mathbb{Q})$ with the property that $\Sigma_X \cdot v = 0$ for any $v \in V$ (by negative deffiniteness of Γ there is a unique such element). The element Σ_X represents the rational Seifert surface of the sublink of L corresponding to X. The *Alexander grading* A_X with respect to X is defined as

$$2A_X[K, E] = K(\Sigma_X - X) - (\Sigma_X - X)^2 + \min_{I \subset E}\{K(I) + I^2\} - \min_{I \subset E}\{(K + 2X^*)(I) + I^2\}.$$

where X^* represents the Poincaré dual of X. For being able to keep track of these Alexander gradings after surgery on some of the link components of L we need to introduce some maps $N_X\colon \mathbb{CF}(Y_\Gamma) \to \mathbb{CF}(Y_\Gamma)$, and later keep track of how these maps change after the surgery. This part I will not explain in the report.

We are in the process of writting up the results, and will in the near future put a preliminary version on arXiv.