Scientific Report
Finsler Chords on Fiberwise Convex Hypersurfaces

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In the framework of the ESF activity entitled 'Contact and Symplectic Topology' I was awarded a grant amounting to a maximum of 5190 EUR, which includes travel costs up to a maximum of 390 EUR, to spend 12 weeks hosted by Professor L. Kozma from the University of Debrecen in Hungary. This grant has reference number 4784.

1 Purpose of the visit

A *symplectic manifold* (W, ω) is a smooth manifold W equipped with a closed non-degenerate 2-form ω. We consider the cotangent bundle T^*M of a n-dimensional closed connected Riemannian manifold (M, g). The cotangent bundle T^*M with the standard Liouville form $\lambda = pdq$ leads to a symplectic manifold $(T^*M, d\lambda)$. Consider a hypersurface Σ in T^*M which is *fiberwise starshaped* with respect to the origin, i.e. $\Sigma_x := \Sigma \cap T^*_x M$ is strictly star-shaped with respect to $0_x \in T^*_x M$. A *contact form* on Σ is a 1-form η on Σ with $\eta \wedge (d\eta)^{n-1} > 0$ everywhere. An example is the 1-form $\lambda|_\Sigma$. The previous contact form η determines generally a *contact structure* on Σ, the oriented hyperplane field $\xi := \ker(\eta) \subset T\Sigma$. In our case this shall be $\xi_\Sigma := \ker(\lambda|_\Sigma)$. Choose a smooth function $H : T^*M \to \mathbb{R}$ which is fiberwise homogeneous of degree one such that Σ is a regular level set of H, a so-called *energy hypersurface*. The *Hamiltonian vector field* X_H is defined by $d\omega(X_H, \cdot) = dH(\cdot)$. The contact form $\lambda|_\Sigma$ determines the unique *Reeb vector field* R on $T\Sigma$ by

$$d(\lambda|_\Sigma)(R, \cdot) \equiv 0, \quad \lambda|_\Sigma(R) \equiv 1.$$

The associated flow is called the *Reeb flow* φ_R of R. One can show that the Reeb flow φ_R of $\ker(\lambda|_\Sigma)$ is a reparametrization of the flow $\varphi_H|_\Sigma$ of X_H

restricted to Σ. A *Reeb chord* is a flow line of φ_R.

Question (A version of the Arnol′d Chord Conjecture, [4]). *Is it true that for any two points $q, q' \in M$, there exists a Reeb chord from Σ_q to $\Sigma_{q'}$?*

There is a special class of fiberwise starshaped hypersurfaces, so-called fiberwise convex hypersurfaces. One way to get its hand on those is to start with a Finsler base manifold (M, F), where F is the Finsler structure on M. This structure leads to a Hamiltonian function on T^*M whose regular level sets are fiberwise convex hypersurfaces in T^*M. Finsler chords on fiberwise convex hypersurfaces are special examples of Reeb flows. I would like to provide qualitative and quantitative results concerning Finsler chords on fiberwise convex hypersurfaces in the cotangent bundle.

Professor L. Kozma from the University of Debrecen in Hungary is a leading expert for Finsler geometry. For instance, he worked on Morse theory for Finsler manifolds, what is of direct interest to me. Work on the generalization of certain Riemannian constructions to the symmetric Finsler and general Finsler situation will strongly involve methods of his area of expertise. Very kindly, he invited me to work with him in Debrecen.

2 Description of the work carried out during the visit

Professor Kozma was responsible for a very nice and stimulating working environment. Discussions concerning my topics and his help in finding articles were a great support. Also other members of the geometry departement of the University of Debrecen supported me very nicely with specific articles.

The people of the geometry departement were very interested in my topic. I gave three talks in the geometry seminar on my research topic. An introductory talk, establishing the link from the basics to my questions, then in a second one, I tried to give the idea how Morse homology can be used to deduce geometrical information about a manifold. The third talk covered concrete results obtained in the past.

A couple of interesting discussions leaded to further insights and understandings of my questions. In addition, I was asked for help on non-directly related research questions.

3 Description of the main results obtained

3.1 Fiberwise convex hypersurfaces

Let (M, F) be a closed connected Finsler manifold. The Finsler structure F leads to a Minkowski norm F_q on every tangent space T_qM – sometimes also called an asymmetric norm –,

$$\begin{array}{rccl} F_q\colon & T_qM & \longrightarrow & \mathbb{R}, \\ & (q,v) & \longmapsto & F(q,v)\,. \end{array}$$

see for example Shen [9] for an exposition of Finsler geometry. In particular, F_q is a convex function since it is an asymmetric norm, meaning since $F_q(v) \neq F_q(-v)$ in general. According to Álvarez Paiva [3], one can consider the dual normed vector space (T_q^*M, F_q^*), where

$$\begin{array}{rccl} F_q^*\colon & T_q^*M & \longrightarrow & \mathbb{R}, \\ & (q,p) & \longmapsto & F_q^*(p) := \sup_{v\in T_qM} \{|p(v)| \mid F_q(v) \leq 1\}\,. \end{array}$$

Then, he defines a Hamiltonian function $H_F\colon T^*M \to \mathbb{R}$ by

$$\begin{array}{rccl} H_F\colon & T_q^*M & \longrightarrow & \mathbb{R}, \\ & (q,p) & \longmapsto & H_F(q,p) := F_q^*(p)\,. \end{array}$$

Level sets of H_F are fiberwise convex since level sets of convex functions are convex sets. Therefore, for any regular value α of H_F the related level set $H_F^{-1}(\alpha)$ is a fiberwise starshaped hypersurface of T^*M. So, given a closed connected Finsler manifold one can construct fiberwise convex hypersurfaces in the associated cotangent bundle.

Let $\Sigma \subset T^*M$ be a fiberwise convex hypersurface. Consider the (unique) connected disc $D_q \subset T_q^*M$ which satisfies $\partial D_q = \Sigma_q$, $q \in M$ and $0_q \in D_q$. Observe that the disc D_q must not be strictly convex. At least if Σ is fiberwise strictly convex, one can construct via a method of Hofer–Zehnder [6] a Finsler structure on the base manifold M.

3.2 Reeb chords on convex energy hypersurfaces via Morse homology

Generally, we want to understand the time spectrum of a fiberwise starshaped hypersurface $\Sigma \subset T^*M$ of the cotangent bundle T^*M by interpreting it as a contact manifold. In particular, we are interested in the special case of fiberwise convex hypersurfaces. Define the time (or the reduced action) of a path $\gamma\colon [0,1] \to T^*M$ of Sobolev-(1,2) type by

$$\mathcal{T}(\gamma) = \int_\gamma \lambda\,, \quad \gamma \in W^{1,2}([0,1], T^*M).$$

Consider the set – the time spectrum of Σ –

$$\mathcal{S}\left(\Sigma, q, q'\right) := \left\{\mathcal{T}\left(\gamma\right) \mid \gamma \text{ a path on } \Sigma \text{ from } \Sigma_q \text{ to } \Sigma_{q'} \text{ solving } \dot{\gamma} = R(\gamma)\right\},$$

where the points $q, q' \in M$ are arbitrarily chosen, the number $\mathcal{T}(\gamma)$ is the time needed by the Reeb chord γ to go from Σ_q to $\Sigma_{q'}$ and R is the Reeb vector field of the contact manifold $(\Sigma, \lambda|_\Sigma)$. Further, we are interested in the *counting function*

$$\mathrm{CF}_{q,q',\Sigma}(T) = \#\left\{\tau \in \mathcal{S}\left(\Sigma, q, q'\right) \mid \tau \le T\right\}. \tag{1}$$

If one fixes a time T, then the counting function gives the number $\mathrm{CF}_{q,q',\Sigma}\,(T)$ of Reeb chords starting in the fiber Σ_q and ending in some point of $\Sigma_{q'}$ before or with the time T.

Let (M, F) be a closed connected Finsler manifold. Define $L_F = \frac{1}{2}F^2$ to be the associated Lagrangian. Recall that the critical points of the (Finsler-) energy functional $\mathcal{E}_F(q) = \int_0^1 L_F(q(t), \dot{q}(t))dt$ are in fact (Finsler) geodesic segments, see [9]. It is the idea to use information and results about Finsler geodesic segments on a Finsler manifold M to get our hands on the time spectrum of Reeb chords on fiberwise convex energy hypersurfaces in the cotangent bundle of M. Our Lagrangian L_F is time-independent, so according to [1], we do the following assumptions on L_F:

(L1) There exists a $\ell_0 > 0$ such that

$$\nabla_{vv} L_F(q, v) \ge \ell_0 \cdot \mathbb{I}$$

for every $(q, v) \in TM$.

(L2) There exists a $\ell_1 \ge 0$ such that

$$\begin{aligned} |\nabla_{vv} L_F(q, v)| &\le \ell_1, \\ |\nabla_{qv} L_F(q, v)| &\le \ell_1(1 + |v|), \\ |\nabla_{qq} L_F(q, v)| &\le \ell_1(1 + |v|^2), \end{aligned}$$

for every $(q, v) \in TM$.

Note that by $|\cdot|$ we denote the norm on the fibers of TM defined by some chosen Riemannian metric g on M. The two conditions above do not really depend on the metric chosen, varying the metric results in taking other constants ℓ_0 and ℓ_1. The strong convexity assumption (L1) on L_F implies that (via the Legendre transform as the pullback of the Hamiltonian vector field) there exists a smooth vector field Y_{L_F} on TM. In terms of this vector field, we can state the next assumption:

(L0) Every solution $q\colon [0,1] \to M$ of Y_{L_F} is non-degenerate, meaning that the differential of the time-one integral map of Y_{L_F},

$$T_{(q(0),\dot{q}(0))}TM \to T_{(q(1),\dot{q}(1))}TM$$

maps the vertical subspace at $(q(0),\dot{q}(0))$ into a subspace having intersection (0) with the vertical subspace at $(q(1),\dot{q}(1))$. (The splitting of TTM is given by the corresponding Levi-Civita connection.)

Via the Legendre transform we can define a Hamiltonian function $H_F\colon T^*M \to \mathbb{R}$. Suppose further that H_F satisfies the following requirements.

(H1) $dH_F(q,p)[p\frac{\partial}{\partial p}] - H_F(q,p) \geq h_0|p|^2 - h_1$ for some constants $h_0 > 0$ and $h_1 \geq 0$, and

(H2) There exists a $h_2 \geq 0$ such that

$$|\nabla_q H_F(q,p)| \leq h_2(1+|p|^2),$$
$$|\nabla_p H_F(q,p)| \leq h_2(1+|p|),$$

for every $(q,p) \in T^*M$.

The Hamiltonian H_F defines a smooth vector field X_{H_F} whose integral flow (which are defined on the time interval $[0,1]$) shall be denoted by $\varphi^t_{H_F}$. We will be interested in the set Sol_{H_F} of solutions $x\colon [0,1] \to T^*M$ such that $x(0) \in T^*_q M$ and $x(1) \in T^*_{q'}M$ for two fixed points $q,q' \in M$.

(H0) Every solution $x \in Sol_{H_F}$ is non-degenerate, meaning that the image of the vertical subspace of $T_{x(0)}T^*M$ by $D\varphi^1_{H_F}(x(0))$ has intersection (0) with the vertical subspace of $T_{x(1)}T^*M$.

As indicated above, L_F shall satisfy (L0)–(L2). In addition, we assume that its Legendre transform H_F satisfies (H0)–(H2). Then, still according to Abbondandolo–Schwarz [1], we know that the Morse homology groups $\mathrm{HM}_k\left(\{CM_*(\mathcal{E}_F), \partial_*(\mathcal{E}_F, g)\}; \mathbb{F}\right)$ of $\mathcal{E}_F$ (where g is the already chosen Riemannian metric) is isomorphic to the Floer homology groups $\mathrm{HF}_k(H_F, q, q'; \mathbb{F})$. In particular, note that the functional $\mathcal{E}_F$ satisfies the Palais-Smale condition. We refer to Abbondandolo–Schwarz [2]. Also therein, the authors establish an isomorphism from $\mathrm{HM}_k\left(\{CM_*(\mathcal{E}_F), \partial_*(\mathcal{E}_F, g)\}; \mathbb{F}\right)$ to the singular homology groups $\mathrm{H}_k(\Omega^1_{q,q'}M, \mathbb{F})$. The space $\Omega^1_{q,q'}M$ is the space of Sobolev-(1,2) type paths $\gamma\colon [0,1] \to M$ satisfaying $\gamma(0) = q$ and $\gamma(1) = q'$. Finally, directly from the results in [1, 2], we obtain the family of isomorphisms

$$\mathrm{HF}_k(H_F, q, q'; \mathbb{F}) \longrightarrow \mathrm{H}_k(\Omega^1_{q,q'}M, \mathbb{F}) \tag{2}$$

for almost all pairs $(q, q') \in M \times M$. After passing to sub-complexes, we get

$$\mathrm{HF}^a_k(H_F, q, q'; \mathbb{F}) \longrightarrow \mathrm{H}_k(\Omega^{1,a}_{q,q'}M, \mathbb{F}) \tag{3}$$

for $a \in \mathbb{R}$ not in the action spectrum of H_F and almost all pairs $(q, q') \in M \times M$. By $\mathrm{HF}^a_k(H_F, q, q'; \mathbb{F})$ we think of the homology group of the through a action bounded solutions of Hamilton's equations associated with H_F; by $\Omega^{1,a}_{q,q'}M$ we understand the space of length bounded Sobolev-(1,2) type paths: $\gamma\colon [0,1] \to M$ satisfying $\gamma(0) = q$ as well as $\gamma(1) = q'$ and $\mathcal{L}_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))dt \le a$.

Finally, we can conclude that if we have homological information about the Finsler manifold (M, F), this yields via the isomorphisms (3) directly to quantitative results on Reeb chords of the fiberwise convex hypersurface $\Sigma = H_F^{-1}(\alpha)$ (with α a regular value).

An application. Let $\Sigma \subset T^*M$ a fiberwise convex hypersurface.

Proposition 3.1. *Let $q \in M$. If $\pi_1(M, q)$ has finite order $k \in \mathbb{N}$, then for every $q' \in M$ there exist at least k Reeb chords x_ℓ, $\ell \in \{1, ..., k\}$, from Σ_q to $\Sigma_{q'}$ satisfying the time bound*

$$\mathcal{T}(x_\ell) \le (1 + \lambda_F)\,\mathrm{diam}(M, F)\,.$$

This proposition follows from Propositions 3.3 and 6.2 of Shen–Zhao [8], see also Gromov [5]. By λ_F we denote the reversibility of F. (The reversibility of F is defined as follows: $\lambda_F := \sup_{(x,y)\in SM} F(x, -y)\,, SM := \bigcup_{x\in M} \{y \in T_xM \mid F(x, y) = 1\}$.)

3.3 Finsler geodesic segments

Knowing Finsler geodesic segments which serve as – in the best case different – representatives of (singular) homology classes lead via the isomorphisms (3) directly to Reeb chords on fiberwise convex hypersurfaces. Therefore, we investigated the article [7] by A. Nabutovsky and R. Rotman whether their results on geodesic segments on closed connected Riemannian manifolds generalize to closed connected Finsler manifolds. In certain situations, their constructions are done by homological means which implies that those can be used directly to deduce results on Reeb chords. We add that the results considered in this report are counting Reeb chords with multiplicities.

We can confirm that all results stated in [7] generalize to the general Finsler situation. And because they are considered with multiplicities we do not get contradictions with existing results.

The main result is the following theorem.

Theorem 3.2. *Let (M, F) be a closed connected n-dimensional Finsler manifold of diameter $d = \operatorname{diam}(M)$. For every two (not necessarily distinct) points $q, q' \in M$ and every positive integer k there are at least k distinct geodesics connecting q and q' of length $\leq 4nk^2 d$.*

Another results which is related to Proposition 3.1:

Theorem 3.3. *If the fundamental group of a closed connected Finsler manifold (M, F) of diameter $d = \operatorname{diam}(M)$ is either infinite or finite of order $\geq k$, then for every pair of points $q, q' \in M$ and every k there exist at least k geodesics connecting q and q' of length $\leq kd$ that represent different path homotopy classes.*

Also this theorem can directly be used to deduce the existence of k Reeb chords satisfying the time bound $\leq \ell d$ for $\ell \in \{1, ..., k\}$.

4 Projected publications/articles resulting or to result from your grant

It is planned to write a publication with full details concerning the results of the stay in Debrecen during the coming January 2015.

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