Scientific Report: Exchange Visit to Barcelona

Andrew Brooke-Taylor

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Purpose of the Visit

The purpose of this visit was to work with Professors Joan Bagaria and Carles Casacuberta, both of the University of Barcelona, on problems at the interface between set theory and algebraic topology.

Description of Work Carried Out

The large cardinal axiom *Vopěnka's Principle* has found important applications in category theory, and indeed is known to be equivalent to a variety of structural statements about certain "nice" categories — *locally presentable* and *accessible* categories. On the other hand, there are also a range of statements about such categories which are known to be equivalent to a certain weakening of Vopěnka's Principle, known naturally enough as *Weak Vopěnka's Principle.* It is not known whether Weak Vopěnka's Principle is strictly weaker than Vopěnka's Principle, and the strongest known lower bound on its consistency strength is the existence of a proper class of measurable cardinals, far below Vopěnka's Principle. There is thus a considerable range in the large cardinal hierarchy where Weak Vopěnka's Principle could lie, and pin-pointing its consistency strength would be of great interest.

One of the ways in which measurable cardinals tie in to category-theoretic results is by means of group theory: there is a measurable cardinal if and only if there is an uncountable cardinal κ such that there is a non-trivial group homomorphism from $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$ to $\mathbb{Z}$. Indeed if such cardinals κ exist, the least will be measurable. This example arises frequently in the area, and so getting to grips with possible variants seemed a good approach to trying to raise the lower bound on the strength of Weak Vopěnka's Principle . In particular, there are other known results using strongly compact and supercompact cardinals to prove similar statements about groups. These were originally proven by category theorists using the associated measures and ultraproducts in a very "hands-on" way. As a first approach, we therefore considered how to streamline these proofs from a more set-theoretic viewpoint, with the hope of then being able to more easily extend the results.

Description of Main Results Obtained

One known connection between large cardinals and group theory is through *radicals.*

Definition 1 *For abelian groups X and A and a cardinal κ, we define*

$$R_X(A) = \bigcup\{\ker(f) : f \in \mathrm{Hom}(A, X)\}$$

and

$$R_X^\kappa(A) = \sum\{R_X(B) : B \subseteq A \wedge |B| < \kappa\}.$$

Here $\mathrm{Hom}(A, X)$ denotes the set of group homomorphism from A to X, as is standard in category theory.

In his paper "On reduced products of abelian groups" (*Rendiconti del Seminario Matematical della Universit di Padova,* tome 73, 1985, pp 41–47) Dugas proved the following result.

Theorem 2 (Dugas 1985) *If κ is a strongly compact cardinal, and X is an Abelian group of cardinality less than κ, then $R_X = R_X^\kappa$.*

His proof uses an ultraproduct of the $<\kappa$-generated subgroups of A over a fine measure on $\mathcal{P}_\kappa(A)$; much of the work is then in checking that everything works nicely. We were able to prove the result using the elementary embedding definition of a strongly compact cardinal, which led to a much more streamlined proof.

Another major outcome of the research was related to a published claim of Rosicky, that for any strongly compact cardinal κ, any theory T for a λ-ary language $\mathcal{L}$ with $\lambda \leq \kappa$, and any κ-directed diagram $\mathcal{D}$ of models of T, if there is a T-model B that is the colimit of $\mathcal{D}$ in the category of models of T, then B is also the colimit of $\mathcal{D}$ in the category of $\mathcal{L}$-structures. Once again, the argument was very "hands on", using atomic diagrams and explicit ultraproducts with a strong compactness measure, and once again, we were able reformulate the argument using elementary embeddings to give a much clearer, cleaner argument. However, contemplating this proof soon after the exchange visit ended led us to realise that there is a simple counterexample to the claim! Namely, let $\mathcal{L}$ be the language with a single binary relation $<$, let T be the $\mathcal{L}$-theory of total orders with a maximum element, and let $\mathcal{D}$ be the diagram of successor ordinals less than κ with the usual inclusions. Then the ordinal κ is the colimit in the category of $\mathcal{L}$-structures, but $\kappa + 1$ is the colimit in the category of models of T. Since finding this example with have isolated the flaw in our version of the argument, and have been able to modify the statement of the claim to make the argument a valid proof. However, we are yet to properly isolate the source of the problem in the original argument, involving as it does unfamiliar results from category theory, and so this work is ongoing.

Future Collaboration

As mentioned above, the work is ongoing. It is hoped that we will be able to meet in 2010 to continue the work, either in Barcelona or in Bristol.

Publications

The results of our research are not yet at a publishable stage. However, it seems reasonable to think that they may form part of a future paper.