Visit funded by ESF Exchange Grant: Final report

David Asperó*

1 Final report

I have visited the Kurt Gödel Research Center (*KGRC*) in Vienna, Austria, from 10 April to 10 May 2010. This visit has been funded by an ESF Exchange Grant. The main purpose of my visit was to continue previous work with Prof. Sy-David Friedman, and also previous work with my former student Dr. Miguel Ángel Mota.

With Friedman we have been working mostly on the revision and simplification of our joint paper [A-F]. In that paper, assuming $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, we build a totally proper partial order with the $\aleph_2$–chain condition and forcing the existence of a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ by a parameter–free formula. Now we have fixed some technical problems in the original proof (detected by a referee) and have also been able to remove some superfluous elements in it. Besides that we have been looking at the problem of forcing, in a general ZFC context, the existence, for a given strong limit singular cardinal λ of uncountable cofinality, of a well–order of $H(\lambda^+)$ definable in $\langle H(\lambda^+), \in\rangle$ from a well–order of $H(\lambda)$. By a result of Shelah ([S2]), under this assumption there is a nice representation of subsets of λ by countable sets of ordinals in $(2^\lambda)^+$, together with a parameter in $H(\lambda)$. We have been looking at how to use this representation for our purposes, still without success.

With Mota we have been working on our method for forcing with symmetric systems of elementary substructures as side conditions. This is proving to

*ICREA (Institució Catalana de Recerca i Estudis Avançats) and Universitat de Barcelona

be a rather fruitful method for constructing forcing notions which are proper (in particular, they preserve ω_1), have the $\aleph_2$–chain condition (in particular they also preserve all cardinals above ω_1) and force $2^{\aleph_0}$ to be large (larger than $\aleph_2$). In our joint work this month we have come up with a simplification of previous constructions using this method. This simplification turns out to make the method more powerful. Indeed, with it we have been able to produce a model of the forcing axiom FA(Γ) together with $2^{\aleph_0}$ being κ for many possible values of κ,[1] where Γ is the class of all those partial orders which have size $\aleph_1$ and which have the (slight) strengthening of Shelah's properness which (for the moment) we have dubbed "finite properness": $\mathbb{P}$ is *finitely proper* in case for every large enough $H(\theta)$ and every finite set $N_0, \ldots N_m$ of countable elementary substructures of $H(\theta)$ containing $\mathbb{P}$, if $N_0 \cap \omega_1 = \ldots = N_m \cap \omega_1$, then for every $p \in \mathbb{P} \cap \bigcap_{i \leq m} N_i$ there is a condition q extending p such that q is $(N_i, \mathbb{P})$–generic for all $i \leq m$. This forcing axiom implies Martin's Axiom at ω_1 (MA_{ω_1}) as well as most failures of Club Guessing following from the Bounded Proper Forcing Axiom (BPFA).[2] This result settles, almost completely (but not quite, at least for the moment), the interesting question whether the forcing axiom for the class of all proper forcings of size $\aleph_1$ – let us call it PFA(ω_1) – implies $2^{\aleph_0} = \aleph_2$. However it is not clear whether this forcing axiom implies such strong failures of Club Guessing as $\neg\mho_n$, for some $n \geq 2$ (these failures of Club Guessing do follow from PFA(ω_1)). Similarly, we have seen that the reflection principle MRP implies that not every proper poset of size $\aleph_1$ is in the above class Γ. Also, using a variation of our construction we have built a model of Moore's *measuring* together with $2^{\aleph_0} = \kappa$ for many possible values of $\kappa > \aleph_2$. Measuring – which is the assertion that for every sequence $\langle A_\delta : \delta < \omega_1 \rangle$, if each A_δ is a closed subset of δ, then there is a club $C \subseteq \omega_1$ such that for each $\delta \in C$, either a tail of $C \cap \delta$ is contained in A_δ or a tail of $C \cap \delta$ is disjoint of A_δ – is the maximal failure of Club Guessing, implies $\neg\mho_2$ (the strongest form of $\neg\mho_n$), follows from MRP and from BPFA, and does not seem to follow from the above forcing axiom PFA(Γ) or even from PFA(ω_1).

With Friedman, Mota, and Dr. Marcin Sabok (who was also visiting the *KGRC* with *ESF* funding) we have been working on several questions regarding the notion of α–properness.[3] For example we have proved, for any

[1]In particular κ can be any $\aleph_n$ ($n \geq 2$).

[2]Before our work during this month we did not know if it is consistent that MA_{ω_1} holds together with significant (strong) failures of Club Guessing and with $2^{\aleph_0} > \aleph_2$.

[3]This is the strengthening of properness in which there are generic conditions for all

pair of indecomposable countable ordinals $\alpha < \beta$, that the forcing axiom for the class of α–proper forcings does not imply even the bounded forcing axiom for the class of β–proper forcings. The central body of the proof is showing that α–proper forcing preserves a certain kind of weak Club Guessing we call *thin β–weak Club Guessing*. Part of these results is actually implicit in a remark in Shelah's Proper Forcing book [S1] (without proof). We have found a clean proof of this remark and have extended it to some extent. In the same context, Sabok has observed that there is a an $<\omega_1$–proper forcing which does not embed regularly in any forcing of the form σ–closed $*$ ccc, and he is working on the problem of characterizing those $<\omega_1$–proper forcing that do embed in such a forcing. At some point in our joint work we decided to focus on the problem whether the bounded forcing axiom for the class of α–proper forcing (for some $\alpha \geq \omega$) decides the size of the continuum. In this context we have proved that the conjunction of $\neg\mho_2$[4] and the assertion that the structure $\langle H(\omega_1), \in, NS_{\omega_1}\rangle$ is a Σ_1–elementary submodel of $\langle H(\omega_2), \in, NS_{\omega_1}\rangle^{V^{Coll(\omega_1,\omega_4)}}$ does imply $2^{\aleph_0} \leq \aleph_2$. The proof of this implication draws heavily on Caicedo–Veličković's coding for proving the existence of a very simply definable well–order of $H(\omega_2)$ under BPFA ([C-V]).

I have also held several conversations with Dr. Vincenzo Dimonte, another postdoc at the KGRC, about large cardinal axioms in the region of an elementary embedding from $L(V_{\lambda+1})$ into itself, and even an elementary embedding from the universe into itself (in ZF). However, no specific work has come out of these conversations yet.

Finally, I gave a 2–hour talk at the weakly KGRC set theory seminar. The title of my talk was "On Π_2 maximality and CH", and in it I presented recent joint work with Paul Larson and Justin Moore.

2 Future plans

With Friedman we will continue working on the problem of forcing well–orders of $H(\lambda^+)$ definable over the structure $\langle H(\lambda^+), \in\rangle$ for λ being a strong

N_i whenever $(N_i)_{i<\alpha}$ is any tower of models containing $\mathbb{P}$. $(N_i)_{i<\alpha}$ is a tower if, for all $\beta < \alpha$, $(N_i)_{i\leq\beta} \in N_{\beta+1}$, and $N_\beta = \bigcup_{\gamma<\beta} N_\gamma$ if in addition β is a limit ordinal. Also, $\mathbb{P}$ is $<\omega_1$–proper if it is α–proper for all $\alpha < \omega_1$.

[4]This statement was referred to in the previous paragraph. It is the assertion that for every sequence $\langle A_\alpha : \delta < \omega_1\rangle$, if each A_δ is a clopen subset of δ, then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ a tail of $C \cap \delta$ is either contained in or disjoint from A_δ.

limit singular cardinal, focusing on the case $cf(\lambda) > \omega$, but with an eye on the case of countable cofinality as well. In this respect, we will try to get a model in which there is a λ which is a limit of countably many supercompact cardinals and such that there is a well–order of $H(\lambda^+)$ definable over $\langle H(\lambda^+), \in\rangle$.[5]

With Mota we are writing a paper with the results we have obtained this last month. This will be a sequel to [A-M]. We will probably present our constructions in a fairly general way. The purpose of such generality will be to provide a neat uniform way of separating instances of the forcing axiom for the class of finitely proper posets of size $\aleph_1$ together with $2^{\aleph_0}$ large; for example it seems we can build a model of $2^{\aleph_0} = \kappa$ together with the forcing axiom for the class of those partial orders in Γ which in addition preserve (for instance) a given "strong" Weak Club Guessing sequence. The corresponding model satisfies then Weak Club Guessing. In our work together we have also considered the problem of forcing various dichotomies for (ω_1–generated) ideals consisting of countable subsets of ω_1, together with $2^{\aleph_0}$ being large. We have not succeeded yet, but we want to work in this direction in the near future.

With Friedman, Mota and Sabok we are planning to eventually write a paper with the work we have done so far and with possible future results. For us, the main problem in this area is undoubtedly the problem whether the bounded forcing axiom for the class of ω–proper forcings implies $2^{\aleph_0} = \aleph_2$. This seems to be a difficult and very interesting problem because all known results of the implication BPFA $\Rightarrow 2^{\aleph_0} = \aleph_2$ use, in some form or another, a forcing which will definitely not be ω–proper in general. In fact, all such proofs use essential consideration of the forcing for adding some given instance of MRP (a reflecting sequence, in Moore's terminology). Therefore it seems that any proof of the implication FA(ω–proper) $\Rightarrow 2^{\aleph_0} = \aleph_2$ (without additional hypotheses) will need completely new coding techniques. We will certainly continue working on this problem.

[5]There are large cardinal situation which prohibit the existence of such well–orders: If there is a non-trivial elementary embedding from some $L(V_{\lambda+1})$ into itself, then there is no such well–order of $H(\lambda^+)$.

References

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[S1] Saharon Shelah, *Proper and Improper Forcing*, Perspectives in Mathematical Logic, Springer, Berlin, 1998.

[S2] Saharon Shelah, *PCF without choice*