Scientific Report

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"On various reflection principles and their generalizations"

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I visited Prof. Veličković at University of Paris 7 from Apr.12 to Jun.18 (2 months). The purpose of my visit was to discuss with Prof. Veličković about various reflection principles on $\mathcal{P}_{\omega_1}(\lambda)$ such as the stationary reflection principle (SR), the semi-stationary reflection principle (SSR) and the Fodor-type reflection principle (FRP). In particular we aimed to make the relationship among these reflection principles clearer by examining their consequences on combinatorial principles.

SR has been studied extensively by many set theorists, but SSR and FRP have not been studied as well as SR. SSR is a weakening of SR and is equivalent to that every ω_1-stationary preserving poset is semi-proper. It was shown in [3] that SSR is strictly weaker than SR. FRP is shown in [1] and [2] to be equivalent to many reflection principles on topological spaces. FRP is implied by Axiom R, which is a strengthening of SR, and it is known that FRP does not imply SSR.

In my research proposal I suggested the following open questions among other things:

- Does SSR implies the failure of $\square(\lambda)$ for every regular cardinal $\lambda \geq \omega_2$?

- Does SSR implies the singular cardinal hypothesis (SCH)?

- Does FRP implies the failure of $\square(\lambda)$ for every regular cardinal $\mu \geq \omega_2$?

(It is known that FRP implies SCH.)

During my stay at Paris, we discussed these questions and could solve the above problems:

Theorem 1.   *(1)* SSR *implies the failure of* $\square(\lambda)$ *for every regular cardinal* $\lambda \geq \omega_2$.

  (2) SSR *implies* SCH.

  (3) FRP(λ) *is consistent with* $\square(\lambda)$ *for any regular cardinal* $\lambda \geq \omega_2$.

In particular, it turned out that FRP differs from SR and SSR in that it is consistent with $\square(\lambda)$.

We could find a new difference between SR and SSR, too. Weiss [4] introduced combinatorial principles (κ, λ)-TP and (κ, λ)-ITP for a regular cardinal $\kappa \geq \omega_2$ and a cardinal $\lambda \geq \kappa$. These principles are generalizations of characterizations of strongly compact and supercompact cardinals, due to T. Jech and M. Magidor. In fact the following holds:

- For an inaccessible cardinal κ,

  - κ is strongly compact if and only if (κ, λ)-TP holds for every $\lambda \geq \kappa$,
  - κ is supercompact if and only if (κ, λ)-ITP holds for every $\lambda \geq \kappa$.

In [4] the following are proved:

- (κ, λ)-ITP implies (κ, λ)-TP.

- PFA implies (ω_2, λ)-ITP for every $\lambda \geq \omega_2$.

- (ω_2, λ)-TP implies the failure of CH.

We investigated the relationship between reflection principles and these new combinatorial principles. We obtained the following results:

Theorem 2. *(1)* SSR *together with Martin's axiom for Cohen forcing implies that* (ω_2, λ)-TP *holds for every* $\lambda \geq \omega_2$.

(2) SR *together with Martin's axiom for Cohen forcing implies that* (ω_2, λ)-ITP *holds for every* $\lambda \geq \omega_2$.

(3) SSR *together with Martin's axiom for Cohen forcing does not imply* (ω_2, λ)-ITP *for any* $\lambda \geq \omega_3$.

Thus we found that the difference between SSR and SR is similar as those between strongly compact and supercompact cardinals.

We are planning to publish our results containing Thm. 1 and 2 above.

We do not know whether the assumption of "Martin's axiom for Cohen forcing" in Thm. 2 (1) and (2) can be weakened to "the failure of CH". This is one of our future works. Moreover I suggested to discuss generalizations of various reflection principles on $\mathcal{P}_{\omega_1}(\lambda)$ to those on $\mathcal{P}_\kappa(\lambda)$. This remains to be our future work, too.

During my stay at Paris, we could also discuss many other problems, and I am very pleased that I could visit Paris. I thank the ESF research networking program "New frontiers of infinity" so much for supporting my visit.

References

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[4] C. Weiss, *Subtle and Ineffable Tree Properties*, Dissertation an der Fakultät für Mathematik, Informatik und Statistik der Ludwig-Maximilians-Universität München.