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Scientific Report
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1   Purpose of the visit

The project concerns different points of view about posets. Generalization of Hausdorff's Theorem on scattered linear orders and its possible extension to partial orders.

Let us say that a partial order P is *scattered* if the partial order does not contains a copy of the rational linear order $\mathbb{Q}$. The first well-know result on scattered linear order is due to Hausdorff [H].

Recall that a poset satisfies *FAC* (*Finite antichain Condition*) whenever every set of pairwise incomparable elements is finite. A poset $\langle P, \preceq \rangle$ is an *augmentation* of $\langle P, \leq \rangle$ if: $x \leq y$ implies $x \preceq y$ for any $x, y \in P$. Abraham and Bonnet shown in [AB] that the class of scattered posets satisfying the FAC, is the closure of well-quasi-orders, under the following operations: inverse order, lexicographic sum along a well-quasi-order, and augmentation.

A linear ordering L is κ-*dense* whenever for every $a < b$ in L, the interval (a, b) is of cardinality κ. A poset P is κ-*scattered* whenever P does not contains a κ-dense linear sub-ordering. (The Hausdorff case corresponds to the case where $\kappa = \aleph_0 = |\mathbb{Q}|$.)

M. Džamonja and K. Thompson [DT], found a complete classification of κ-scattered linear orderings, and showed that any augmentation of a FAC κ-scattered posets is κ-scattered. Also in the same work, they obtain a partial classification of FAC κ-scattered posets.

Another possibility to extend the notion of scattereness was introduced again by Sierpinski, with the chains, denoted by η_α or by $\mathbb{Q}_\kappa$ where $\kappa = \aleph_\alpha$. (Again, the Hausdorff case corresponds to $\kappa = \aleph_0 = |\mathbb{Q}|$.) A poset P is $\mathbb{Q}_\kappa$-*scattered*

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whenever P does not containsa $\mathbb{Q}_\kappa$-dense linear sub-ordering. In the same work [DT], the authors showed that any augmentation of a FAC $\mathbb{Q}_\kappa$-scattered posets is $\mathbb{Q}_\kappa$-scattered and a partial classification of FAC κ-scattered posets was obtained.

She shall see also, that for the embeddding relation, the notion of better-quasi-orderings and well-quasi-orderings appears, using results of Laver [L] and Todorcevic [T].

References

[AB] U. Abraham and R. Bonnet: Hausdorff's theorem for posets that satisfy the finite antichain property. *Fund. Math.* 159 (1999), no. 1, 51–69.

[BP] R. Bonnet and M. Pouzet: Linear extensions of ordered sets. *Ordered sets (Banff, Alta., 1981)*, NATO Adv. Study Inst. Ser. C: Math. Phys. Sci., 83, Reidel, Dordrecht-Boston, Mass., 1982, pp. 125–170,

[DT] M. Džamonja and K. Thompson: A poset hierarchy, *Cent. Eur. J. Math.* **4**(2), (2006), pp :225–241

[H] F. Hausdorff: Grundzüge einer Theorie der geordneten Mengen, *Math. Ann.* 65 (1908) 435–505.

[L] R. Laver, On Fraïssé's order type conjecture, *Ann. of Math.* (2) 93 (1971) 89–111.

[T] S. Todorčević: Very strongly rigid Boolean algebras, *Publ. Inst. Math. (Beograd)* 27 (41) (1980), 267–277.

2   Description of the work carried out during the visit

During my visit, Uri Abraham and James Cummings came to work with Mirna Džamonja and me, to improve the technics that Uri and me developed in [AB]. We found the main key to improve the two notions of scatteredness. This was based on the introduction of an equivalence relation on a scattered FAC poset, in such a way that the original poset is a lexicographic sum of "more simple" posets over a scattered chain. Later K. Thompson came, and, with Mirna, we improve in a deep way some of the results.

Another trick was the introduction of κ-fat poset to clean some notions, and to simplify some proof of [DT]. A poset P is κ-*fat* whenever the interval (a, b) is of cardinality κ for every $a < b$ in P. (For instance, the lexicographix sum over $\mathbb{Q}$ of antichains of cardinality $\aleph_1$ is $\aleph_1$-fat but $\aleph_1$-scattered.)

The last topic was to to apply R. Laver's result [L], who solved a stronger version of the "Fraïssé conjecture". More precisely the class of σ-scattered linear

orderings is a better-quasi-ordering, and thus a well-quasi-ordering for the embedding relation. We recall that a *σ-scattered* linear ordering is a countable union of scattered linear ordering. A *well-quasi-ordering* (*wqo*) is a class with no strictly decreasing sequence nor infinite antichain. The notion of *better-quasi-order* (*bqo*) was introduced by C. St. J. A. Nash-Williams, in 1968, and *any bqo is wqo*.

A direct consequence of Laver's result, is the fact that *the class of $\aleph_0$-scattered linear orderings is a wqo*, and that *the class of $\aleph_1$-scattered linear orderings is also a wqo*.

Note that there is an $\aleph_1$-dense linear ordering which is σ-scattered. In fact, for any infinite cardinal κ, there is a κ-dense linear ordering which is σ-scattered.

Using a result of S. Todorčević [T], for a given regular cardinal κ, to each $A \subseteq \kappa$, we can associate a $\aleph_1$-dense linear ordering $L(A)$ of cardinality κ such that: $A \subseteq B$ if and only if $L(A)$ is embeddable in $L(B)$. Notice that $L(A)$ is $\aleph_2$-scattered. So *the class of $\aleph_2$-scattered linear ordering is not a wqo*.

3 Description of the main results obtained.

We prove first the following result.

Theorem 1.*Let $\mathcal{B}_0$ be the class of chains of cardinality less than κ. Let $\mathcal{L}_0^\kappa$ be the closure of $\mathcal{B}_0$ over lexicographic sum with index set either a WQO poset, the inverse of a WQO poset, or an element of $\mathcal{B}_0$.*

Then $\mathcal{L}_0^\kappa$ is the class of κ–scattered linear orders.   □

A class $\mathcal{G}_0$ of linear orderings is *reasonable* if and only if $\mathcal{G}_0$ contains a nonempty ordering and is closed under reversals and restrictions.

Given a reasonable class $\mathcal{G}_0$ of linear orderings, the closure $\mathrm{cl}(\mathcal{G}_0)$ of $\mathcal{G}_0$ is the least class of posets which contains $\mathcal{G}_0$ and is closed under the operations:
- Lexicographic sum with index set either a WQO poset, the inverse of a WQO poset, or an element of $\mathcal{G}_0$.
- Augmentation.

Note that, in the following, (R2) is a (direct) consequence of the non-trivial result (R1).

(R1) Let $\mathcal{G}$ be the class of FAC posets such that every chain is in $\mathcal{G}_0$. Then $\mathcal{G} \subseteq \mathrm{cl}(\mathcal{G}_0)$.

(R2) Let $\mathcal{G}_0$ and $\mathcal{G}$ be as above, and assume in addition that:
(1) $\mathcal{G}_0$ contains all well-orderings, and is closed under lexicographic sums with index set in $\mathcal{G}_0$.
(2) $\mathcal{G}$ is closed under augmentations.

Then $\mathcal{G} = \mathrm{cl}(\mathcal{G}_0)$.

As an application of Theorem 1 and (R2), we obtains the following result.

Theorem 2. *Let $\mathcal{B}_0$ be the class of chains of cardinality less than κ. Let $\mathcal{L}^\kappa$ be the least class of posets which contains $\mathcal{B}_0$ and is closed under the operations:*
- Lexicographic sum with index set either a WQO poset, the inverse of a WQO poset, or an element of $\mathcal{B}_0$.
- Augmentation.
    Then $\mathcal{L}^\kappa$ is the class of all FAC κ–scattered posets. $\qquad\square$

The second application of (R2) concerns $\mathbb{Q}_\kappa$–scatteredness.

Theorem 3. *Let $\mathcal{L}_0^{\mathbb{Q}_\kappa}$ be the class of $\mathbb{Q}_\kappa$–scattered linear orders. Let $\mathcal{L}^{\mathbb{Q}_\kappa}$ be the least class of posets which contains $\mathcal{L}_0^{\mathbb{Q}_\kappa}$ and is closed under the operations:*
- Lexicographic sum with index set either a WQO poset, the inverse of a WQO poset, or an element of $\mathcal{L}_0^{\mathbb{Q}_\kappa}$.
- Augmentation.
    Then $\mathcal{L}^{\mathbb{Q}_\kappa}$ is the class of all FAC $\mathbb{Q}_\kappa$–scattered posets. $\qquad\square$

Tha classification of $\mathbb{Q}_\kappa$–scattered linear orders seems impossible. Indeed the fact to be $\mathbb{Q}_\kappa$–scattered is not absolute, as we can see in the next result.

Theorem 4. (1) *Let κ be an infinite cardinal. Then the property of being a κ-scattered linear ordering is upwards absolute to cardinal preserving extensions.*

(2) *Let* (CH) *hold. Then there exist a $\mathbb{Q}_{\aleph_1}$–scattered linear ordering L and a forcing poset which adds no reals, such that L is not $\mathbb{Q}_{\aleph_1}$–scattered in the generic extension.* $\qquad\square$

4 Future collaboration with host institution.

We state only few questions related to this subject that remain open.

A. Linearization of scattered posets

Recall that a *linearization* $\langle P, \preceq \rangle$ *of a poset* $\langle P, \leq \rangle$ is a a linear order $\langle P, \preceq \rangle$ satisfying: $x \leq y$ implies $x \preceq y$ for any $x, y \in P$. (So a linearization is an augmentation.) Note that Bonnet and Pouzet proved that any scattered poset has a scattered linearization. So, the following is open.

Question 1. Let κ be cardinal.
    1. Does any κ–scattered poset has a κ–scattered linearization?
    2. Assume $\kappa^{<\kappa} = \kappa$. Does any $\mathbb{Q}_\kappa$–scattered poset has a $\mathbb{Q}_\kappa$–scattered linearization?

B. Embeddablity between scattered posets

For posets P and Q, the fact that P is embeddable in Q, means that there is $f : P \to Q$ such that: $x \leq y$ if and only if $f(x) \leq f(y)$ for $x, y \in P$. In what follows we compare posets for the embedding relation.

Let κ be a cardinal and ρ be an ordinal. We denote by $\mathcal{D}_\rho^\kappa$ the class of scattered posets of cardinality at most κ and of antichain rank at most ρ.

R. Laver showed that $\mathcal{D}_1^\kappa$ (scattered linear orderings) is bqo and thus wqo. We try to extend this result to the class of FAC scattered orderings. For instance, that the following holds (the proof is not written yet):

The class $\mathcal{D}_\omega^{\aleph_0}$ has an infinite antichain and a strictly decreasing sequence.

So, the we ask the following question.

Question 2.
1. Is $\mathcal{D}_n^{\aleph_0}$ bqo or wqo for every finite $n \geq 2$?
2. In particular is $\mathcal{D}_2^{\aleph_0}$ bqo or wqo?
3. The same questions (1) and (2) replacing $\aleph_0$ by an infinite cardinal κ.

5   Projected publications/articles resulting or to result from the grant

Article <u>submitted</u> to *Transactions of Amer. Math. Soc.* at the end of my visit :

U. Abraham, R. Bonnet, J. Cummings, M. Džamonja and K. Thompson : *A Scattering of Orders*, 26 pages,

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