FINAL REPORT ON CAST EXCHANGE GRANT 3178

The CAST exchange grant 3178 was awarded to fund two visits (3+4 weeks) of David Martínez Torres (I.S.T. Lisbon) to Eva Miranda (Universitat Politécnica de Catalunya, Barcelona), aimed at developing the project "Hamiltonian versus weakly Hamiltonian actions in Symplectic and Poisson geometry and applications to normal forms of integrable systems".

Due to health complications of David Martínez Torres, the second scheduled visit had to be postponed and finally cancelled. Unfortunately, this has reduced the original scope of the project and slowed down its progress. In this report we will describe the research initiated during the visit that took place form 11/21/2010 to 12/11/2010.

1. Work carried out during the visit

The study of symmetries is a central topic in symplectic geometry. The best possible scenario is that of Hamiltonian actions. i.e. when the infinitesimal symplectic symmetries have Hamiltonian potentials fitting into an equivariant momentum map (or equivalently, when the action relates the Poisson geometry on the symplectic manifold to the linear Poisson geometry of the dual of the Lie algebra). If the Hamiltonian potentials do not fit into an equivariant momentum map, then we are in the case of weakly Hamiltonian actions (and the link Poisson geometry on the symplectic manifold to the linear Poisson geometry of the dual of the Lie algebra is lost). Finally, if no Hamiltonian potential exists we simply have a symplectic action.

There has been much activity on the search of constrains for Hamiltonian actions and the difference between Hamiltonian, weakly Hamiltonian and symplectic actions in symplectic geometry. From a more global perspective, this can be seen as an analysis of the infinite dimensional group of symplectic diffeomorphisms and its subgroup of Hamiltonian diffeomorphisms.

Our work during this visit has been centered on exploring properties of the Lie algebra of the group of Poisson diffeomorphism of a Poisson manifold, and of its Lie subalgebra Hamiltonian vector fields.

More explicitly, Delzant has proved [3] that in a compact symplectic manifold, if a semisimple Lie algebra acts in a effective manner then it must be of compact type.

- We have explored whether in a compact Poisson manifold any finite dimensional semisimple subalgebra of the algebra of Hamiltonian vector fields is of compact type. The methods we used are similar to those of Delzant. From a more modern perspective they amount to studying the presence of certain real analytic functions associated to any infinitesimal (weakly) Hamiltonian actions [5].

In a compact symplectic manifold there is no difference between Hamiltonian and weakly Hamiltonian actions. This is because there exists a universal momentum map. Also the group of Hamiltonian diffeomorphisms is simple [1]. To have no distinction between Hamiltonian and weakly Hamiltonian actions in a compact Poisson manifold, we need to ask for unimodularity (existence of Hamiltonian invariant volume forms) [7].

- For a compact unimodular regular Poisson manifold we have looked at the question of the simplicity of the Lie algebra of Hamiltonian vector fields. This has been done by analyzing the relation between three subalgebras of the algebra of smooth functions: the derived ideal (of functions), the leafwise exact volume forms (turned into functions using a Hamiltonian invariant volume form) and the functions with zero mean w.r.t. any Hamiltonian invariant volume form.

2. Results

We have been able to obtain new information on the Lie algebra of Hamiltonian vector fields of a compact Poisson manifold. Namely we have proved the following:

(1) If a compact Poisson manifold carries an effective weakly Hamiltonian action of a semisimple Lie algebra, then the Lie algebra must be of compact type.
(2) There exists examples of regular unimodular Poisson manifolds, such that the derived ideal of the Lie algebra of Hamiltonian vector fields has finite codimension (it is not difficult to see that for symplectic fibrations this codimension is infinite).

The construction of the manifolds in (ii) is built on both the description of unimodular regular Poisson manifolds of codimension one [4], and on some computations of foliated cohomology in [6, 2]. At this point we believe that some more time needs to be devoted to a more systematic study of the Lie algebra of unimodular regular Poisson manifolds of codimension one. Together with the material described above, this should lead to the corresponding publication.

3. Additional activities

During the course of the visit David Martínez Torres delivered two invited seminars at Universitat Politécnica de Catalunya, Barcelona. As a consequence of one of them, a collaboration with Ignasi Mundet i Riera (Universitat de Barcelona) was initiated, its topic being the study of the fundamental group of the group of diffeomorphisms of coadjoint orbits. This work is expected to give rise to a publication in a very near future.

References

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