

Foundational and Philosophical Implication of the Inner Model Hypothesis

ESF activity:
“New Frontiers of Infinity:
Mathematical, Philosophical and Computational Prospects”
Final Report
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1 Purpose of the visit

My visits to the Kurt Gödel Research Center for Mathematical Logic (KGRC) in Vienna took place during following weeks: 16–30 October 2011, 13–27 November 2011, 4–18 December 2011. Their purpose was to enable me to investigate, from both a philosophical and a mathematical point of view, the *Inner Model Hypothesis* (IMH), an axiomatic approach due to Sy D. Friedman, who formulated it in the context of the so called *Hyperuniverse Program*.¹

2 Description of the work

During my stay at the KGRC my efforts have been mostly directed at analyzing the overall program of which the Inner Model Hypothesis is a component, in the attempt of making its general philosophical presuppositions and foundational implications explicit.

The *Hyperuniverse Program* can be described as follows. Its aim is to enlarge the axiomatic system ZFC so as to provide a framework within which new theoretic truths can be discovered (among the latter one wishes to include sentences that settle questions independent from ZFC). The strategy one adopts to this purpose is to create a context (i.e. the hyperuniverse) in which different pictures of the set theoretic universe can be compared, whereby the comparison is expected to deliver insights on the features that preferred universes of sets should display, i.e. to evoke axioms that express *criteria* for “favoring”, on legitimate grounds, certain universes to others.

¹The *Inner Model Hypothesis* is the statement: *If a statement ϕ of the language of set theory without parameters holds in an inner universe of some outer universe of V , then it already holds in some inner universe of V .* See [Fri06] for details.

Starting from the preferred universes of the hyperuniverse, one then proceeds to enlarge the realm of the statements that are to be regarded as true in V , guided by the assumption that the first-order properties of the preferred universes (possibly including solutions to independent questions) are shared by such universes and V (an assumption that basically rests on the downwards Löwenheim–Skolem theorem – in fact the members of the hyperuniverse are countable transitive models of ZFC).

This being, briefly sketched, the *Hyperuniverse Program*, the issues why it makes sense to expect that final solutions of ZFC-independent statements can be found, and to search for them by the strategy suggested by the advocate of the Hyperuniverse Program, naturally arise, and call for analysis. One should make clear as well what notion of set theoretic truth emerges from the approach at issue, and how it relates to already expressed positions on truth in set theory. Worth of being cautiously considered are also the mathematical consequences which one may be led to by subscribing to the *Hyperuniverse Program* (especially by invoking the IMH as a criterion for preferred universes), and the way these may affect one's foundational views. The research carried out at the KGRC consisted exactly in identifying, addressing and analyzing the issues indicated. The work took place in the form of intensive daily discussions together with Sy D. Friedman, followed by individual investigation.

3 Results

Due to joint deep analysis with Sy D. Friedman, clear views could be developed as to how formulate and present:

- the *Hyperuniverse Program* as an overall proposal for going beyond independence results in set theory,
- the philosophical assumptions underlying it, especially with respect to the notion of truth in set theory,
- the different possibilities that are there for implementing the program, and their mathematical implications.

As far as the last point is concerned, it is worth remarking that in suggesting new axioms for set theory, which is done by the advocate of the *Hyperuniverse Program*, one not only is committed to the search of principled motivations for them. One also has to cohere with a corpus of already existing and accepted mathematical results, which the new axioms (and their consequences) should shed some light on or, at least, not irremediably conflict with. This seems not to happen, though, if a statement like the *Inner Model Hypothesis* is adopted as a new axiom for set theory. In fact it

contradicts the existence of large cardinals in the universe (but not the consistency of the statement that they exist in inner models). How one should regard this result, whether as discarding the IMH as a plausible axiom candidate or as witnessing that a reformulation of the basic ideas underlying it is needed, is an issue that has been intensively considered by Sy D. Friedman and me. Discussed has been as well the related question how the role of large cardinal hypotheses in contemporary set theory should be properly understood. An alternative appears to be there between regarding the latter as true-in- V propositions, on the one hand, and, on the other, taking them as consistent assertions within a hypothetical framework – which happens when one proves that set theoretical statements are consistent starting from the assumption that the existence of some large cardinal is so. The question whether all large cardinals should be regarded as on a par, from both the epistemic and the mathematical point of view, has been addressed too. This led us to make a difference, under both respects, between “small” and “large” large cardinals, and to examine how the *Inner Model Hypothesis* could be so reformulated as to be compatible with the assertion that small large cardinals exist in the universe.

The ideas developed during my stay at the KGRC have been partly presented by both Sy D. Friedman and me at the *4th Workshop on Philosophy of Mathematical Practices*, “Maximalist and Minimalist Perspectives on Infinity”, held in Sevilla on November 7-8, 2011.² A joint paper with Sy D. Friedman (title: *The Hyperuniverse Program*), meant as both an introduction to the program and a critical appraisal of it in the broader context of the foundational debate in contemporary set theory, is in preparation. Once ready, it will be submitted to the *Bulletin of Symbolic Logic*.

4 Future collaborations and publications

A further visit of mine to the KGRC has been planned in January, in order to complete the joint paper *The Hyperuniverse Program*, which, if accepted, will be the publication directly resulting from the research supported by the ESF within the activity “New Frontiers of Infinity: Mathematical, Philosophical and Computational Prospects”.

References

- [Fri06] S. D. Friedman. Internal consistency and the inner model hypothesis. *Bulletin of Symbolic Logic*, 12(4):591–600, 2006.

²See <http://institucional.us.es/phum2594/settheoryworkshop.html>