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ITGP - Exchange Grant - 3675
Report

22 February, 2012

1 Introduction

This Exchange Grant contemplated the visit of Dr. Pedro Lopes to Professor John Barrett at the University of Nottingham for 12 weeks starting on 03 November 2011. Pedro Lopes thanks the European Science Foundation and the programme "Interactions of Low-Dimensional Topology and Geometry with Mathematical Physics (ITGP)" for this grant.

2 The Purpose of the Visit

The purpose of this visit was to study the correspondences set forth by Kreimer between Feynman diagrams and knots ([2]). Professor Louis H. Kauffman from the University of Illinois at Chicago, USA, is also involved in this project.

Given a Feynman diagram, Kreimer obtains a link by clasping the loops from a base of loops of the diagram, and via skeining, he obtains a knot which he claims carries information concerning the counterterms of the renormalization associated to the original Feynman diagram. In particular, he claims the knot obtained in this process does not depend on the choice of the base of loops. Moreover, he points out that certain L loop Feynman diagrams yield torus knot of type $(2, 2L-3)$ and counterterms given by $\zeta(2L-3)$. All this is done based on illustrative examples, there does not seem to be definitions and proofs of statements available.

The goal of this visit was to try to put these ideas on a firm mathematical setting. In particular to come up with formal definitions for the association of links and knots to Feynman diagrams, and to try to prove some of the statements, or to prove statements in not so general situations. For instance, we would have liked to have obtained a formal proof for the independence from the base of loops.

It should be said that although we did not come up with a formal definition for the association referred to above, we feel this project is still worth pursuing and we outline below the work developed and possible new approaches to this problem.

3 Description of the work carried out during the visit and results

3.1 Introduction

Some time was dedicated to studying possible operational definitions (at the graphical level) for obtaining the clasping of loops once the loops were assigned. For instance, we tried the rule that loops with adjacent edges should clasp along the adjacent edge. Although this rule was successful in simple instances, it failed in the general case.

We next looked into the contributions of the integration variables (some of the internal momenta) of the integral (associated to the Feynamn diagram under study) to this problem. We realized that some choice of integration variables gave rise to factors in the integrand depending on the difference of momenta in adjacent loops, in two's. We wonder if this was what led Kreimer to the notion of clasping of loops. In fact, the dependence on the difference of consecutive momenta expresses a reciprocity between consecutive loops which at a graphical/topological level is well illustrated by the clasping of two loops. Moreover, the other choice of integration variables that we seem to find in the literature, does not give rise to these factors depending on the difference of momenta in two consecutive loops. In this other instance ([3]) the inner loops do not clasp among themselves but the last one clasps with each one of the others, keeping up with a requirement of least number of crossings as possible and also expressing the fact that there is some interdependence in the integrations after all. Anyway, no operational definition at the graphical level emerged from these observations.

We elaborate on these remarks in the next Section.

3.2 Discussion

For a planar Feynman diagram (which are the ones we work with here) a standard set of loops is obtained in the following way. Remove the vertices and edges of the Feynaman diagram from the plane where it was drawn. A number of discs is left behind. The boundaries of these discs constitute the "standard loops" for the indicated Feynman diagram. Loops are systematically oriented counterclockwise.

3.2.1 The graphical tentatives

Kreimer indicates the following steps from a Feynman diagram to the corresponding knot ([2], chapter 5). View Figure which shows the procedure applied to the diagram known as the tetrahdron. Given a Feynman diagram with L loops and without the external legs (i), consider a base of loops (ii), clasp the loops (iii), skein as $L-1$ times at crossings stemming from different clasps (obtaining (iv), where we only show the knot obtained from the smoothings in the skeinings).

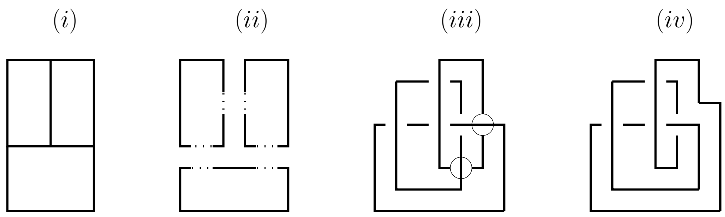

Figure 1: Four moments (i) through (iv) from the Feynman diagram without external legs to the knot associated to it: the case of the tetrahedron. The dotted lines flag the claspings. The circles flag where the skeinings take place. The knot obtained is $\widehat{\sigma_1^3}$ a.k.a., 3_1 a.k.a., the trefoil knot.

Here the rule of clasping loops with adjacent edges works well. Let us see what happens adding another loop to the top portion of the diagram, view Figure 2, where we again choose standard loops.

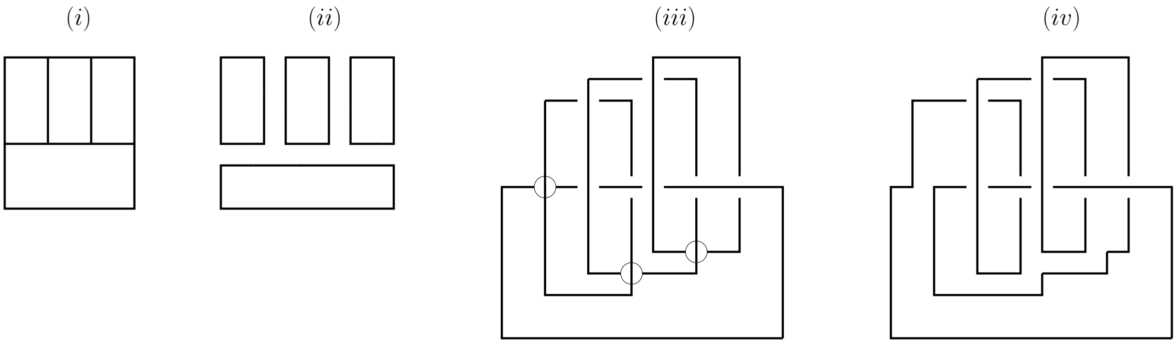

Figure 2: Four moments (i) through (iv) from the Feynman diagram without external legs to the knot associated to it: four loop diagram and standard basis. The knot obtained is $\widehat{\sigma_1^5}$ a.k.a., 5_1.

Now the same diagram, view Figure 3, but choosing non-standard loops. In this instance the clasping of adjacent loops does not work anymore. In particular because it does not yield the desired knot at the end of the skeining. In the next Subsections we try to illustrate why a different clasping is needed here.

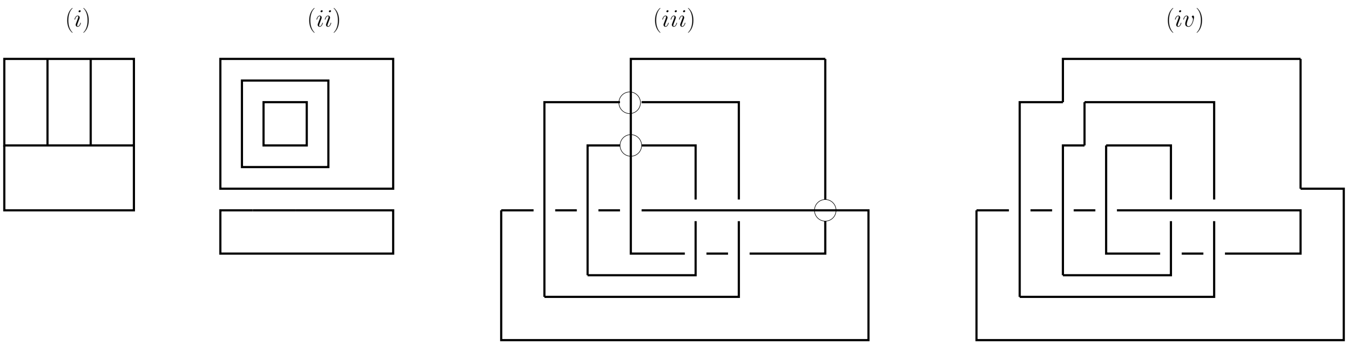

Figure 3: Four moments (i) through (iv) from the Feynman diagram without external legs to the knot associated to it: four loop diagram and non-standard basis. The knot obtained is $\widehat{\sigma_1^5}$ a.k.a., 5_1.

Other Feynman diagrams ([2], page 194 and 196) were studied in search of an operational definition for the clasping from the graphical point of view but without success.

3.2.2 Example with the "standard" loops

Consider Figure 4 which depicts a "ladder diagram" with four loops. In [4], this is Figure 2 with the number of loops $L = 4$. The corresponding Feynman integral for generic L is, again according to [4]:

$$\int \cdots \int \frac{d^4 r_1 \dots d^4 r_L}{(r_1 - r_2)^2 (r_2 - r_3)^2 \cdots (r_{L-1} - r_L)^2 r_L^2} \left\{ \prod_{i=1}^{L} (p_1 + r_i)^2 (p_2 - r_i)^2 \right\}^{-1}$$

which in our $L = 4$ case boils down to

$$\int\int\int\int \frac{d^4 r_1 d^4 r_2 d^4 r_3 d^4 r_4}{(r_1 - r_2)^2 (r_2 - r_3)^2 (r_3 - r_4)^2 r_4^2} \cdot$$
$$\cdot \frac{1}{(p_1 + r_1)^2 (p_2 - r_1)^2 (p_1 + r_2)^2 (p_2 - r_2)^2 (p_1 + r_3)^2 (p_2 - r_3)^2 (p_1 + r_4)^2 (p_2 - r_4)^2} =$$
$$= \int \frac{d^4 r_1}{(p_1 + r_1)^2 (p_2 - r_1)^2} \frac{1}{(r_1 - r_2)^2} \frac{d^4 r_2}{(p_1 + r_2)^2 (p_2 - r_2)^2} \frac{1}{(r_2 - r_3)^2} \cdot$$
$$\cdot \frac{d^4 r_3}{(p_1 + r_3)^2 (p_2 - r_3)^2} \frac{1}{(r_3 - r_4)^2} \frac{d^4 r_4}{(p_1 + r_4)^2 (p_2 - r_4)^2 r_4^2}$$

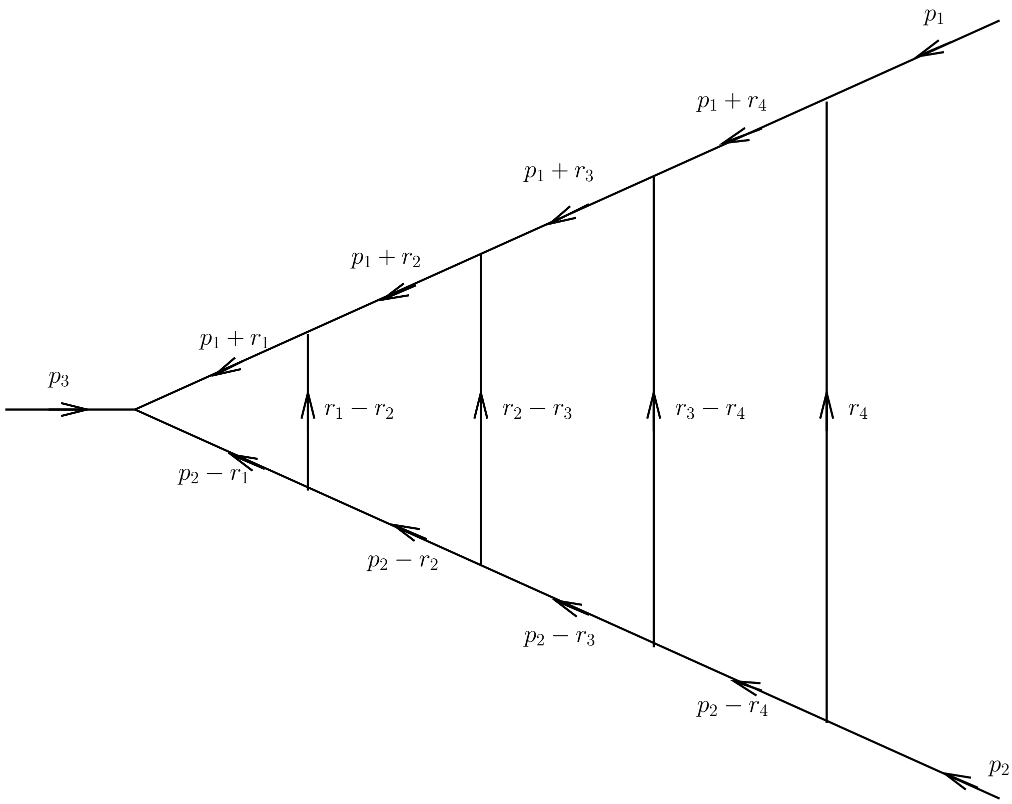

Figure 4: Ladder graph on 4 loops - the r integration variables

At this point we propose the following graphical interpretations (see also the explanation of Figure 3 concerning the choice of momenta in [3], page 120).

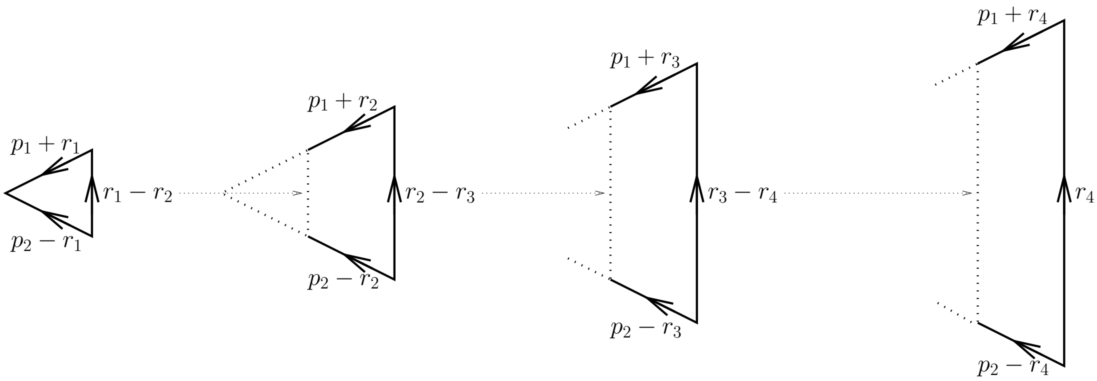

Figure 5: Graphical interpretations - the r integration variables - sketch of the associated loops

Note that each pair of shared edges correspond to a factor of the sort $\frac{1}{r_i - r_{i+1}}$ which expresses mutuality and/or reciprocity between the integrations/loops over r_i and r_{i+1}. Was this what led Kreimer to think of the clasping of loops? We remark that it is not merely the fact that the loops share edges. This sort of phenomenom will not occur with the next choice of integration variables - leading to, according to Kreimer, a different sort of clasping of the loops involved.

The result we extract here is that the standard loops of the ladder diagram and their clasping along adjacent edges seem to illustrate the interdepence of the indicated set of integration variables in the integral associated to this Feynman diagram.

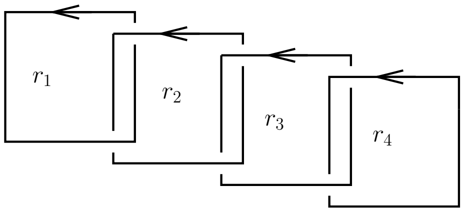

Figure 6: Graphical interpretations - the r integration variables - the shared edges clasp

3.2.3 Example with the non-"standard" loops

Let us now choose a different integration variable. We will fix again $L = 4$ for definiteness, hoping the reader will devise the case for generic L. We set $s_i = r_i - r_{i+1}$, for $i = 1, \ldots, L-1$ and $s_L = r_L$. Consider Figure 7 for the $L = 4$ case. Consider also [3], on page 120, the explanation of Figure 4 concerning the momenta and the construction of the loops.

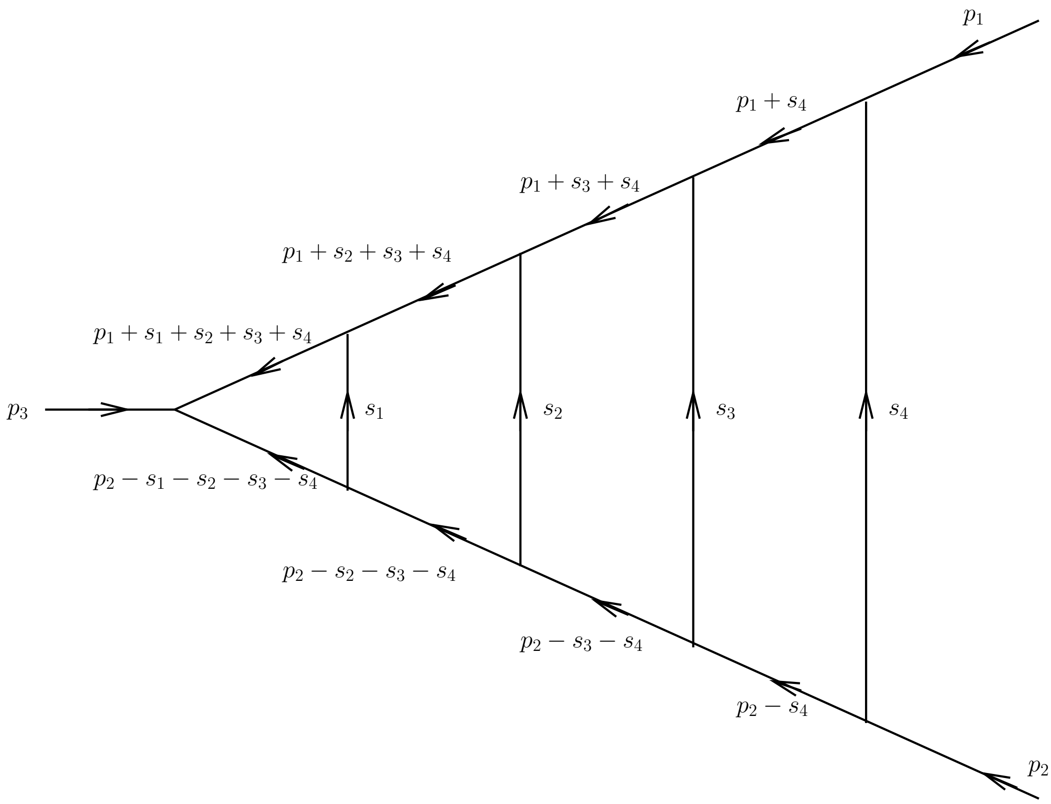

Figure 7: Ladder graph on 4 loops - the s integration variables

The Jacobian of this transformation of variables is 1, so for $L = 4$, the integral associated to this ladder diagram is in the new s variables

$$\cdots = \int \frac{d^4s_1}{(p_1+s_1+s_2+s_3+s_4)^2(p_2-s_1-s_2-s_3-s_4)^2}\frac{1}{s_1^2}\frac{d^4s_2}{(p_1+s_2+s_3+s_4)^2(p_2-s_2-s_3-s_4)^2}\frac{1}{s_2^2}$$

$$\frac{d^4s_3}{(p_1+s_3+s_4)^2(p_2-s_3-s_4)^2}\frac{1}{s_3^2}\frac{d^4s_4}{(p_1+s_4)^2(p_2-s_4)^2s_4^2} =$$

$$= \int \frac{d^4s_1}{[(p_1+s_1+s_2+s_3)+s_4]^2[(p_2-s_1-s_2-s_3)-s_4]^2s_1^2}\frac{d^4s_2}{[(p_1+s_2+s_3)+s_4]^2[(p_2-s_2-s_3)-s_4]^2s_2^2}$$

$$\frac{d^4s_3}{[(p_1+s_3)+s_4]^2[(p_2-s_3)-s_4]^2s_3^2}\frac{d^4s_4}{(p_1+s_4)^2(p_2-s_4)^2s_4^2}$$

Here Kreimer's interpretation seems to be the following (see also his book, [2], Figure 5.5 on page 113). There is no mutuality/reciprocity between pairs of integration variables as in the preceding case. Therefore, the "inner loops" do not clasp. The integral as expressed above develops from left to right and for each of the integration variables there are "external momenta" denoted inside brackets. There is a nested structure in the sense that the integration over s_i has to be done before the integration over s_{i+1} is done. Note that in the preceding case, in the 'middle" of the integration it is not clear whether the integration over s_i has to be done before the integration over s_{i+1}. Finally, there is still an interdependence between all the integration variables which Kreimer expresses by having the last loop clasp with each one of the other loops.

Here are some graphical interpretations.

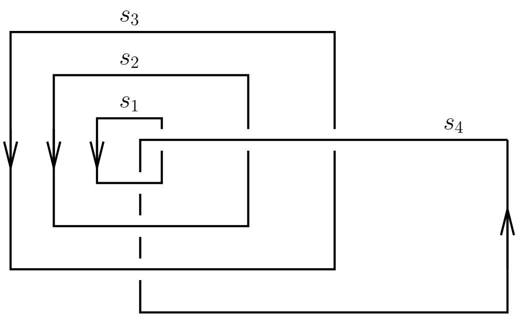

Figure 8: Graphical interpretations - the s integration variables - the last loop clasps with the other ones

The result we extract here is that when associating a set of nested loops to a "ladder diagram" (in the way indicated above) and clasping the outer loop to each of the other loops and allowing no more clasping, we are illustrating the interdependence of this new set of integration variables in the integral associated to this Feynman diagram.

4 Future colaboration with host institution - projected publications

Motivated by the intriguing facts that Kreimer reports like the association between the slashed ladder diagrams on L loops which give rise to counter terms depending on $\zeta(2L-3)$ and also giving rise to the torus knots of type $2L-3$, we feel this topic should be further investigated. In this way we are currently studying Zimmermann's forest formula in the framework of Renormalization Theory ([1]) in order to try to understand the appearance of the counterterms in the more general cases and so by reversing steps perhaps get a better understanding of what the rules for clasping of loops should be. The role of the integration variables is under study too. This is currently taking place at the University of Nottingham as Lopes' visit to Professor John Barrett continues through August 2012. We expect to write an article reporting on our findings.

References

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[4] N. I. Ussyukina, A. I. Davydychev, *Exact Results for Three- and Four-Point Ladder Diagrams with an Arbitrary Number of Rungs*, Phys. Lett. B **305** (1993) 136 - 143