Scientific Report on the Exchange Visit Grant
"Constructive Set-Theoretic Foundations: Induction in Algebra"
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My research during this visit to the Department of Mathematics of Stockholm University upon invitation by Erik Palmgren was centered around the use of Zorn's Lemma in mathematics, especially in algebra and with regard to whether—on the grounds set in [8, 13]—any given invocation of Zorn's Lemma can be replaced by one of Raoult's Open Induction [11].

1. With Simon Huber from Chalmers (who was visiting Stockholm during my own visit) we have studied—upon suggestion by Palmgren—the relationship of the so-called Bourbaki-Witt Principle with Open Induction. The Bourbaki-Witt Principle says that if a map $f : X \to X$ on a chain-complete partial order X is expansive (i.e., $x \leq f(x)$ for every $x \in X$), then f has a fixed point above every $x \in X$. Bourbaki [7] and Witt [14, 15][1] have used this first, to facilite the deductions of Zermelo's Well-Ordering Theorem and/or of Zorn's Lemma from the Axiom of Choice.

   While the Bourbaki-Witt Principle is provable in **ZF**, and in particular without the Axiom of Choice, Bauer and Lumsdaine [5] have recently shown with topos-theoretic methods that the Bourbaki-Witt Principle cannot be proved in any constructive set theory. We have now unveiled that the Bourbaki-Witt Principle can be deduced from Open Induction in (a fragment of) Aczel's constructive variant **CZF** [1, 2, 3] of **ZF**.

2. To prove that the category of abelian groups has enough injective objects one usually uses Zorn's Lemma in the form of Baer's "ideal criterion" for injectivity. In fact, in **ZFA** it is tantamount to the Axiom of Choice that every divisible group is injective [6]. Motivated by recent work [4] on (enough) injective objects in **CZF**, and combining methods from [12, 10], I could reduce the use of transfinite methods to a single invocation of Zorn's Lemma taking place at the very end of the argument. An equational characterisation of injective modules further seems to suggest a dynamical approach.

3. Towards the end of my visit I re-joined my co-author Matthew Hendtlass in the preparation of the final version of [8], for which we still had to clarify severy important points. The content of this paper can be summarised as follows (from the abstract): "In functional analysis it is not uncommon for a proof to proceed by contradiction coupled with an invocation of Zorn's lemma. Any object produced by such an application of Zorn's lemma does not in fact exist, and it is likely that the use of Zorn's lemma is artificial. It has turned out that many proofs of this sort can be simplified, both in form and

[1] Palmgren's PhD student Christian Espíndola helped me with Witt's Spanish.

complexity, with the principle of open induction isolated by Raoult as a substitute for Zorn's lemma. If moreover the theorem under consideration is sufficiently concrete, then a far weaker instance of induction suffices and, with some massaging, one may obtain a fully constructive proof. In the present note we apply this method to Gelfand's proof of Wiener's theorem, producing first a simple direct proof of Wiener's theorem, and then an even simpler constructive proof. With this example in mind we look toward developing a more generally applicable technique."

Last but not least, with Hajime Ishihara, JAIST (who was visiting Stockholm during my own visit), and with Erik Palmgren we have achieved categorical characterisations of the principles of Collection and Strong Collection. This is of particular relevance inasmuch as Strong Collection, one of the distinctive axioms and axiom schemes of Aczel's **CZF** mentioned above, has been conceived by reflection from the interpretation of **CZF** into Martin-Löf's Intuitionistic Type Theory **ITT** [9]; more specifically, (Strong) Collection is related to the choice principle particular to **ITT**.

During my visit to Stockholm I gave a 120 minutes lecture on "A Proof Pattern in Algebra" in the Stockholm Logic Seminar (29 February 2012). At the end of my stay I had the opportunity to join Ishihara and my host Palmgren to the Workshop on Proof Theory and Constructivism held at the Department of Philosophy of Helsinki University on 23 March 2012, where I gave a 45 minutes talk on "Proofs by Induction".

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