# Final Report: $q$-Integrability in String Theory

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**Purpose of Visit:**

To enable deeper and more effective collaboration between host Professor J.L. Miramontes and myself.

**Description of work carried out during the visit and main results obtained:**

The description of work and main results will be benchmarked against the original aims.

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**Aim 1: Construct the TBA for arbitrary $q$**

The first aim will be to construct and investigate and TBA system for the interpolating theory and see whether there are simplification for $q$ a root of unity as suggested by the truncation of the spectrum.

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This goal was largely achieved by another collaboration in a preprint released in August and consequently this aim was replaced by others (see below).

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**Aim 2: Develop the representation theory of the quantum group of centrally extended $\mathfrak{su}(2|2)$ for $q$ a root of unity.**

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This aim was achieved. The first breakthrough was to re-write the S-matrix of the q-deformed theory in a dual kink representation. This was achieved by generalizing the known vertex to interaction-round-a-face of Boltzmann weights of integrable lattice models. We found a way to use this technology in the quantum supergroup. In the new representation the restrictions that appear when $q$ is a root of unity are simple to implement. As a bonus we discovered that the kink representation of the S-matrix was manifestly Hermitian analytic thereby solving a puzzle about how the $q$-deformed theory could be unitary. We also understood how the dual kink representation arises naturally from a careful treatment of the boundary conditions of the symmetric space sine-Gordon therories.

**Aim 3: Understand the physical sheet of the interpolating theory**

The aim here would be to understand which poles can be thought of being due to exchange of bound-states and which can be understood as either anomalous thresholds or are unphysical since they lie off the physical sheet.

This aim was achieved and we now have a complete understand of the analytic structure of the $q$-deformed S-matrix.

**Aim 4: Complete the bootstrap programme for the interpolating S-matrix**

The aim here would be to construct the S-matrix elements of all the bound states of the theory using the bootstrap equations and show that all singularities on the physical strip can be accounted for.

Some progress was made here: we understood how bootstrap equations could be solved by using representation theory alone.

**Aim 5: Understanding the physical interpretation of the $q$ deformation**

We have made a lot of progress on this issue. We have shown using semi-classical methods that the $q$-deformation corresponds to a modification of the symplectic structure on the string world sheet. In fact we found precisely the form of the symplectic structure that interpolates between the string theory and symmetric space sine-Gordon theory.

**Aim 6: Check the S-matrix of the interpolating theory in the semi-classical limit**

This aim was totally achieved. We developed new methods to take the semi-classical limit of the S-matrix and found exact agreement with the classical scattering theory.

**Future Collaboration**

The collaboration between myself and Professor J.L. Miramontes has been enhanced by the visit and will continue into the foreseeable future.

**Projected Publications**

I estimate that 4 publication will result out of work that was either finished or started during the visit. Draft titles:

"Bootstrapping the Magnon S-Matrix"

"Unitarity, Hermitian Analyticity and the $q$-Deformed Worldsheet S-matrix."

"Semi-Classical Limits of the $q$-Deformed Worldsheet S-matrix."

"q-Deformation of String in $AdS_5 \times S^5$ and Symplectic Structures."

**Other Comments**

The original duration of the visit was projected to 6 weeks. In the end due to various administrative constraints I was only able to stay for 5 weeks (I accept that the value of the grant may have to be reduced accordingly).

**Travel Costs:**

As originally stated in the application I travelled by car via the Plymouth-Santander ferry whose cost was more than original estimate of 900 €. I expected only a contribution of 450 € based on the cost of flight and train transfers and I will include quote for these costs when I send in the tickets for the ferry.