FINAL REPORT - ESF EXCHANGE VISIT GRANT

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June 6, 2012

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The main purpose of my stay in the United States in April and May 2012 was a joint research project with Prof. Marek Biskup (UCLA) and Michele Salvi (TU Berlin). Our aim was to analyze the annealed behavior of rescaled local times in the *random conductance model (RCM)*, focusing on the case of uniformly elliptic conductances. However, recent research activity in the community has led us to shift focus to a different, but closely related feature of the RCM, namely the fluctuations of the Dirichlet energy around its mean under certain boundary conditions. With this particular subfield being more time-critical due to a higher degree of competition, we came to the conclusion that it was appropriate engaging this problem before finishing the other. Beyond that, both problems require a deep understanding of the underlying homogenization techniques, making the one quite instructive in view of further analysis of the other, at least from an educational point of view. Let me point out the current state of our joint research in both projects.

1. RCM and random walk among random conductances

Consider the lattice $\mathbb{Z}^d$ and assign to any edge (x, y) connecting two neighboring sites $x \sim y$ a random weight $\omega_{x,y} \in [0, \infty)$, subject to the symmetry condition $\omega_{x,y} = \omega_{y,x}$. The quantity $\omega_{x,y}$ is referred to as *conductance*, whereas its reciprocal is called *resistance*. Denote by $\mathcal{N}$ the canonical base of $\mathbb{Z}^d$, i.e., the set of neighbors of the origin with nonnegative entries. We assume that $(\omega_{x,x+e})_{x\in\mathbb{Z}^d, e\in\mathcal{N}}$ is a family of positive i.i.d. random variables, and we abbreviate $\omega(x, e) = \omega_{x,x+e}$. Define the randomly perturbed Laplacian Δ^ω by

$$\Delta^\omega f(x) := \sum_{y\in\mathbb{Z}^d:\, y\sim x} \omega_{x,y}(f(y) - f(x)). \tag{1.1}$$

This operator is self-adjoint and generates the continuous-time random walk $(X_t)_{t\in[0,\infty)}$ in $\mathbb{Z}^d$, the random walk among random conductances (RWRC).

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2. Asymptotics for local times

2.1 Introduction

We focus on the annealed long-time behavior of the random walk, that is, we average over the field of conductances. We intend to describe the extremal behavior of the random walk in the unlikely sequence of events that it does not leave the time-dependent region $\alpha_t B \cap \mathbb{Z}^d$ up to time t, where $B \subset \mathbb{R}^d$ is some bounded, connected and open set and α_t is a scaling function satisfying $\alpha_t \to \infty$ and $\alpha_t \ll t$ as $t \to \infty$. Consider the local times

$$\ell_t(z) = \int_0^t \delta_{X_s}(z)\,\mathrm{d}s, \qquad z \in \mathbb{Z}^d, t > 0,$$

which measure the amount of time the walker has spent in z up to time t. One of the main motivations for the analysis of local time asymptotics is the *parabolic Anderson model* (i.e., the Cauchy problem for the heat equation) with random conductances and an i.i.d. random potential ξ. To analyse this model in future work, we will have to analyse the expectation of

$$U_t = \exp\{\sum_{z\in\mathbb{Z}^d} l_t(z)\xi(z)\}.$$

A crucial preparation is to derive a *large deviation principle (LDP)* for continuous-space rescaled local times of the walk, that is, to find a scale γ_t and a rate function J such that

$$\Big\langle \mathbb{P}^\omega\Big(\tfrac{\alpha_t^d}{t}\ell_t\big(\,\cdot\,/\alpha_t\big) \approx f^2 \text{ on } B\Big)\Big\rangle \approx \exp\{-\gamma_t J(f^2)\} \quad \text{as } t \to \infty$$

in an appropriate sense. In the equation above $\langle\cdot\rangle$ indicates the expectation with regard to the field of conductances and $f \in H_0^1(B)$ is an L^2-normed function.

2.2 Current state of research project

The annealed behavior of the rescaled local times are governed by a combined "effort" of the conductances and the walk to stay in the growing box ([KSW11]). A first conjecture is that the main contribution to the probability that the rescaled local times resemble a certain deterministic profile f^2 comes from the time-dependent event

$$A_t = \{\omega(z,e) \approx \varphi(z/\alpha_t, e) \text{ for all } z \in \alpha_t B \cap \mathbb{Z}^d, e \in \mathcal{N}\}$$

with an optimally chosen function $\varphi : B \times \mathcal{N} \to (0,\infty)$. Conditioned on A_t, the probability of $\big\{\alpha_t^d t^{-1}\ell_t\big(\,\cdot\,/\alpha_t\big) \approx f^2 \text{ on } B\big\}$ decays exponentially with scale $t\alpha_t^{-2}$ by the Donsker-Varadhan-Gärtner LDP, whereas the probability of A_t itself is of volume scale. The critical box size is thus given by

$$\alpha_t = t^{\frac{1}{d+2}}.$$

We were able to derive an upper bound for exponential decay rate of the non-exit probability from the box with the above approach. This upper bound is given in terms of a variational problem and involves the log-moment generating function of the conductance distribution, implying that the asymptotics potentially depend on all details of that distribution.

However, the proof of this upper bound contains some quite rough estimates and we believe that this approach might not be sensitive enough towards small scale irregularities the field of conductances exhibits with very high probability. Our current conjecture is that the key to a rigorous understanding of the annealed local time asymptotics is a combination of stochastic homogenization techniques on a mesoscopic scale under a local change of measure and subsequent optimization of the global profile of measure changes. This approach is, in contrast to the first one, apt to account for very irregular conductance configurations at a local level.

3. Fluctuations of the Dirichlet energy

3.1 Introduction

The large-scale behavior of the operator Δ^ω has been a central object of study for a long time. Let us have a short look at its associated Dirichlet form. Consider a box $\Lambda_L \subset \mathbb{Z}^d$ centered in the origin with side length $2L$. For a function $f\colon \Lambda_L \to \mathbb{R}$, the Dirichlet form in Λ_L of Δ^ω is defined as

$$Q_L(f) = \sum_{x\sim y,\, \{x,y\}\cap\Lambda_L\neq\emptyset} \omega_{x,y}\big(f(y)-f(x)\big)^2.$$

For any vector $t \in \mathbb{R}^d$, the Dirichlet energy under linear boundary condition t is given by

$$Q_L^t = \inf\Big\{Q_L(f):\ f(z) = t\cdot z \quad \forall z \in \partial\Lambda_L\Big\} = Q_L(t\cdot\psi_L),$$

where $\psi_L = (\psi_L^1, \dots, \psi_L^d)$ and $\psi_L^i\colon \Lambda_L \to \mathbb{R}$ is the unique minimizer of the above variational problem choosing t to be the i-th unit vector. For large L, it is known that

$$Q_L^t \sim Q_L(t\cdot\psi) \sim t^2 L^d A$$

where

$$A = \big\langle \sum_{x\sim 0} \omega_{0,x}|\psi(x)|^2\big\rangle$$

and ψ is the so-called *harmonic coordinate*, the unique mapping $\mathbb{Z}^d \to \mathbb{R}^d$ satisfying $\Delta^\omega\psi = 0$ and $\psi(0) = 0$. The quantity A is referred to as *effective conductance* which accounts for the fact that on large scales Δ^ω behaves in many ways like the continuous Laplacian multiplied with this quantity, compare e.g. [B11]. When considering discrete differential equations on the lattice w.r.t. the elliptic operator Δ^ω, it is therefore of considerable importance to control the fluctuation of Q_L^t.

3.2 Current state of research project

We have established a framework to prove a central limit theorem for the Dirichlet energy Q_L^t which seems likely to admit an extension to a broader class of boundary conditions (e.g., mixed Dirichlet-Neumann) and domains. The first step is to represent the deviation of Q_L^t from its mean by the Doob martingale (indexed by L) we obtain by conditioning on the edges contributing to Q_L^t. Here choosing the right ordering of these edges is of vital importance. Each increment admits a representation in terms of the gradient of $t\cdot\psi_L$. In order to employ the martingale CLT, it is essential to rely on the ergodicity of the environment and replace ψ_L by the shift-covariant full-lattice analogue ψ. Effectively, this requires controlling the expectation of

$$L^{-d}\sum_e |t\cdot\nabla_e(\psi_L-\psi)|^4 = L^{-d}\sum_e |t\cdot\nabla_e(\chi_L-\chi)|^4, \tag{3.1}$$

where the sum includes all edges in the box Λ_L and the correctors χ_L resp. χ are given by

$$\chi_{(L)}(z) = \psi_{(L)}(z) - z.$$

The finite-volume corrector χ_L satisfies

$$\Delta_L^\omega \chi_L = \nabla g_L \qquad \text{on } \Lambda_L \tag{3.2}$$

where the local drift g_L is bounded and supported on Λ_L, and Δ_L^ω denotes the restriction of Δ^ω to Λ_L with zero boundary conditions. Equation (3.2) translates to

$$\nabla\chi_L = \nabla(\Delta_L^\omega)^{-1}\nabla^*[A\nabla\chi_l + g_L] \tag{3.3}$$

where the random operator A is a contraction under the additional assumption of a sufficiently small ellipticity contrast. A suitable uniform ℓ^p-estimate on the random operator $\nabla(\Delta_L^\omega)^{-1}\nabla^*$ implies the uniform estimate

$$\|\nabla\chi_L\|_{\ell^p(\Lambda_L)} \lesssim L^d$$

which yields control of the expectation of (3.1) by uniform p-integrability. Analogous ℓ^p-estimates for the operator $\nabla(\varepsilon - \Delta^\omega)^{-1}\nabla^*$ uniformly in $\varepsilon > 0$ are known as Meyers' estimates. This method has been successfully employed for an optimal variance estimate in a similar approximation of the effective conductance in the recent paper [GO11]. However, in the finite-volume case, the operator under consideration is not of convolution type and a stronger estimate on the corresponding family of kernels in spirit of the Calderón-Zygmund theory seems to be needed.

4. Future collaboration and projected publication

Close collaboration between Marek Biskup, Michele Salvi and myself is going on on a continuous basis, an additional meeting of the three collaborators in Europe is planned in the near future. The aim of this meeting is to completely resolve the remaining open issues and a subsequent submission of the obtained results for publication.

Finally, let me express my gratitude for the support the ESF provided for me in order to realize this research project, which has been and continues to be of great importance for my career and education.

Berlin, June 2012 - Tilman Wolff

References

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