

ESF EXCHANGE GRANT 4045- FINAL REPORT

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PURPOSE OF THE VISIT

The purpose of my visit to the Kurt Godel Research Center was to start a collaboration with Prof. Friedman over the problem of whether or not $\aleph_{\omega+2}$ can consistently satisfy the strong or the super tree property. This topic represents the natural continuation of my doctoral research (see Fontanella [2] and [3]). Indeed, during my Phd I worked on the consistency of the strong and the super tree properties at the cardinals of the form $\aleph_n + 2$ with $n < \omega$ and at $\aleph_{\omega+1}$. All these consistency results were directed towards the more general goal of determining whether is it possible to build a model of set theory where every regular cardinal has the strong or the super tree properties.

DESCRIPTION OF THE WORK CARRIED OUT DURING THE VISIT

Friedman and Halilovic [4] proved that in some forcing extension, $\aleph_{\omega+2}$ has the tree property, assuming the existence of a weakly compact hypermeasurable cardinal. By working on this and related papers, we obtained some interesting results that partially solve the main problem of the proposed project. First we worked on the following theorem by Kanamori.

Theorem: (Kanamori [5]) Assume ρ is an inaccessible cardinal and $\lambda > \rho$ is a weakly compact cardinal, then the forcing iteration $Sacks(\rho, \lambda)$ (i.e. the λ -length iteration of $Sacks(\rho)$ with supports of size $\leq \rho$) produces a model where $\lambda = \rho^{++} = 2^\rho$ and ρ^{++} has the tree property.

By slightly changing the hypothesis, we easily generalized such a theorem to the super tree property and proved the following.

Theorem: Assume ρ is an inaccessible cardinal and $\lambda > \rho$ is a supercompact cardinal, then the forcing iteration $Sacks(\rho, \lambda)$ produces a model where $\lambda = \rho^{++} = 2^\rho$ and ρ^{++} has the super tree property.

Then, we went further and analyzed the following result.

Theorem: (Dobrinen and Friedman [1]) Assume GCH holds and there is a cardinal κ which is weakly compact hypermeasurable, then there is a model of set theory

where κ^{++} has the tree property and κ is still measurable.

By applying minor modifications to the forcing construction used to prove the previous theorem, we succeeded in generalizing even such a result to the super tree property, hence we proved the following.

Theorem: Assume GCH holds and there is a cardinal κ which is supercompact hypermeasurable, then there is a model of set theory where κ^{++} has the super tree property and κ is still measurable.

The next step would be to turn the measurable cardinal κ into \aleph_ω and see whether $\kappa^{++} = \aleph_{\omega+2}$ still has the super tree property in the generic extension.

DESCRIPTION OF THE MAIN RESULTS OBTAINED

The research carried out during the visit was very fruitful. The main result obtained is the following.

Theorem: Assume GCH holds and there is a cardinal κ which is supercompact hypermeasurable, then there is a model of set theory where κ^{++} has the super tree property and κ is still measurable.

Although this theorem provides just a partial answer to the main question of the proposed project, it suggests that a model of the super tree property at $\aleph_{\omega+2}$ can be found. Indeed, we can try the following approach. Suppose W is a model satisfying the conclusion of the previous theorem, namely in W there is a measurable cardinal κ such that κ^{++} has the super tree property. Following the proof of Friedman and Halilovic paper, we want to turn κ^{++} into $\aleph_{\omega+2}$ by preserving the super tree property at κ^{++} — so that $\aleph_{\omega+2}$ would have the super tree property. For that, we can force with a Prikry-type forcing construction with collapses such as the one presented in [4]. At this stage of our investigation we still don't know whether this is going to work, but we expect that we will be able to prove the consistency of the super tree property at $\aleph_{\omega+2}$ by using this or similar forcing constructions.

FUTURE COLLABORATIONS WITH THE HOST INSTITUTION

The Kurt Gödel Research Center offered me some grants to keep investigating this problem in collaboration with Prof. Friedman. With that founding I will be able to work at the institute at least until the end of 2013 and this will allow me to complete the project and to investigate other related open questions.

PROJECT PUBLICATIONS/ARTICLES RESULTING OR TO RESULT FROM THIS GRANT

The main result obtained during the visit is new and can make the object of a publication. However, it provides just a partial answer to the general question of whether $\aleph_{\omega+2}$ can consistently satisfy the super tree property. Therefore, we will write and submit a paper with this result only when we will have fully answer this question. In accordance to the terms and conditions for this Exchange Visit grants, we will acknowledge the support received from the ESF in that paper, and we will forward a reprint to the ESF Secretariat as soon as available.

REFERENCES

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