**Scientific Report ESF Exchange Grant København, University of Copenhagen – Jord Boeijink**
**Noncommutative geometry and quantisation of gauge systems**

This is the final report regarding an exchange grant for J. Boeijink's (M.Sc.) visit to the Noncommutative Geometry group of Ryszard Nest at the University of Copenhagen in Denmark during November 2012.

## Applications of noncommutative geometry to gauge systems and their quantisation

The research of both Ryszard Nest and Jord Boeijink is anchored in noncommutative geometry. This mathematical field was founded by Alain Connes [3] as a generalisation of differential geometry. It has numerous applications in various other fields of mathematics and also in physics.

Boeijink's research focuses on gauge systems and their (strict) quantisation ([9, 10]) using well-known methods from noncommutative geometry. In classical gauge systems the reduced space obtained by singular Marsden-Weinstein reduction is singular as soon as the group of gauge symmetries has fixed points (more precisely, the singular structure is a stratified symplectic space, [11, 12] (see also [8])).

The first aim of Boeijink's current project is to put the singular structure at the classical (reduced) space into noncommutative geometrical data (e.g. constructing objects akin to spectral triples or Dirac operators). The second aim is to subsequently determine the quantum analogue of a stratified structure using methods from noncommutative geometry. These questions will be studied in some particular finite-dimensional examples. Moreover, it is the intention to relate this approach to the work of Huebschmann ([4, 5]), in which the singular structure is visible on a Hilbert space through a so-called co-stratification. However, in this work a quantum observable algebra remains absent.

## Aim of the exchange

The purpose of the visit was to collaborate with Nest on understanding the quantisation of stratified Marsden-Weinstein quotients in some finite-dimensional gauge systems using techniques from noncommutative geometry. Nest was considered to be the ideal person for collaboration on this project because of his enormous expertise on noncommutative geometry and his work on strict quantisation of symplectic spaces ([6, 7]) and manifolds with boundary ([1, 2, 7]).

## Description of the work carried out and main results

The stratified structure on the singular quotient space is related to a stratification on the total space by projection ([8, 11, 12]). Boeijink and Nest looked at how the stratified structure of the total *and* quotient space can be put into a spectral triple, thereby focusing on the role of the Dirac operator. During their work in November Nest and Boeijink noted that on the (compact) total space the singular strata become visible by looking at well-chosen elliptic differential operators of high enough order on the principal stratum (an open and dense submanifold that is always present if the stratification is induced by a proper Lie group action). In particular, one can look at a high enough power of a Dirac operator, if one is present. Details of how the data of the stratification can be put into a spectral triple is still ongoing work.

The stratification on the singular quotient space (the ordinary quotient, not the Marsden-Weinstein one) can, under some conditions, be obtained by restricting to the $G$-invariant part

of a $G$-equivariant spectral triple on the total space. Further properties of this spectral triple, and how the stratification of the quotient can be obtained from it, are still under investigation.

The situations on both the total and the quotient space are very much related to 'boundary' value problems, where the 'boundary' has a higher codimension (see [13, 14]). Boeijink's visit to Copenhagen provided an excellent opportunity to speak with Prof. Gerd Grubb and Dr. Heiko Gimperlein, from whose experience on boundary value problems Boeijink could benefit.

## Future collaboration

Details of the work mentioned above have to be carried out. Future work will also involve the construction of a spectral triple for the singular Marsden-Weinstein quotient and the understanding of how the stratified structure carries over to the quantisation. During the visit the main foundations and directions of the research have been determined. Now that Nest has become familiar with the contents of Boeijink's project, collaboration can be continued by correspondence via e-mail or Skype, when Boeijink has returned to his home institute.

## Project Publications

Boeijink expects to publish a paper as part of his Ph.D.-project. Concerning the collaboration with Nest, the continuation of the work started in Copenhagen will probably result in a joint paper.

## References


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