Set-Theory and Higher-Order Logic: Foundational Issues and Mathematical Developments

Øystein Linnebo, Hannes Leitgeb, and Michael Detlefsen

September 26, 2011

Contents

1 Summary

This summer school on set theory and higher-order logic consisted of four days of tutorials (August 1-4), as well as a subsequent two-day workshop (August 5-6), all hosted by the Institute of Philosophy in London. The goal of this summer school was to provide an interdisciplinary forum in which set theorists and philosophers of mathematics—as well as students of these disciplines—could interact, exchange ideas, and learn from one another. The bipartite structure of the summer school (tutorials, workshop) was designed to promote this goal: for, such structure not only allowed for the presentation of recent research in the workshop component, but also granted participants the opportunity to gain entry to recent discussions in adjacent disciplines via the tutorial component.

The topic of higher-order logic is naturally central to considerations at the intersection of set theory and philosophy of mathematics. For, on the one hand, the axioms of set theory are most naturally rendered by means of higher-order logic. But, on the other hand, set theory has traditionally been conceived as a foundation of mathematics, both in the sense that the set-theoretic axioms function as an arbitrator of mathematical existence and in the sense that the availability of set-theoretic formalization is a condition *sine qua non* on mathematical rigor. A focal question in this area is thus how to understand the second- and higher-order logic employed in the formalization of these foundational axioms. There are important philosophical dimensions to this question, such as whether the notion of a higher-order object (a plurality, a Fregean concept) can be regarded as conceptually prior to the notion of set itself. But there are also deep mathematical facets to this question, such as whether all large cardinal axioms can be understood to follow from higher-order renditions of the set-theoretic axioms (e.g. reflection principles), and how complex the associated higher-order consequence relations may be. See the Scientific Summary (§ 2) for a more detailed description of how the tutorials and workshop talks were responsive to this and other related focal issues; also see the attached reproduction of the Full Program (§ 6).

In terms of the speakers, participants, and organizing partners, the summer school was highly representative of both the European set theory and philosophy of math communities. Six of our eleven speakers hailed from mathematics or logic departments, while five came from philosophy departments or faculties. Further, with the exception of two speakers, all of our speakers currently have positions in European countries, and in total seven European countries were so represented amongst our speakers (see § 4 for details). Similarly diverse was the large group of participants, most of whom are research students at European institutions (cf. § 5 for details). This diversity was aided by two factors which in our estimation contributed measurably to the success of the summer school. The first was the fact that we were able to offer ten 200 GBP stipends to help defray travel costs for students. The second was that the organizing partners were based in three different European countries (respectively, the UK, Germany, France) with projects funded by three different organizations (respectively, European Research Council, Alexander von Humboldt-Stiftung, and L'agence nationale de la recherche), and so were thus able to facilitate the participation of a large group of students from a wide variety of countries and institutions.

2 Description of Scientific Content

Since detailed abstracts for the tutorials and workshop talks are available in the Full Program (cf. § 6), it seems prudent here to give a more thematically-orientated summary of the scientific content of the summer school. Some of the over-arching ideas of the summer school may be collected under the following three general headings: calibrating the strength of higher-order resources (§ 2.1), hierarchies: from type theory to large cardinals (§ 2.2), the metamathematics of set-theoretic independence (§ 2.3).

2.1 Calibrating the Strength of Higher-Order Resources

The subject-matter of mathematical logic is, among other things, a series of inter-related measures of the complexity of definitions and theories, relative to which the strength of definitions or theories with disparate topics may be compared. Several of the tutorials and workshop talks focused on the strength of higher-order resources. One well-known measure of the complexity of theories is that of interpretability strength. This is a partial order of theories induced by the interpretation relation on theories: roughly, one theory is said to interpret another if there are formulas of the interpreting theory relative to which the interpreting theory proves the axioms of the interpreted theory. Canonical examples here the interpretation of arithmetic in set theory and the interpretation of non-Euclidean plane geometries in three-dimensional Euclidean space.

It is a consequence of Gödel's second incompleteness theorem that the higher-order versions of theories like Peano arithmetic and ZFC set-theory have higher interpretability strength than the first-order versions of these theories. However, it is not obvious that the same amount of strength is added in each case; that is, it might have been the case that the interpretability strength of the second-order theory increases exponentially with the interpretability strength of the underlying first-order theory. In his workshop talk, Visser establishes a result which indicates that this eventuality does not occur: rather, there is a precise sense in which the addition of certain higher-order resources adds the same amount of strength regardless of theory in question. In particular, for theories which are similar to Peano arithmetic and ZFC set-theory in the respect that they can code syntax in a specific way (so-called "sequential" theories), Visser gave a proof that a certain higher-order version of a given theory (the so-called "predicative" version) has the same interpretability strength as the first-order version plus its consistency statement (or indeed, as a weak arithmetic plus this consistency statement).

One of the most natural examples of a higher-order notion is that of truth. Indeed, perhaps one of the most well-known theorems of mathematical logic is Tarski's Undefinability Theorem, one version of which says the set of numerical codes for true first-order statements of arithmetic is not a first-order definable set of natural numbers. Since it is easy to see that the set of true statements is second-order definable, this theorem tells us that truth is a properly higher-order notion. Horsten's workshop talk, as well as several points of his tutorial, focused on ways in which the consistency strength (and thus the interpretability strength) of arithmetic theories varies as different kinds of predicates are added that approximate in various ways the truth predicate for arithmetic. For instance, only some of these theories prove that the truth predicate commutes with various logical operators, e.g. that a true conjunction has all and only true conjuncts. Ascertaining the strength of these various theories has been thought by many to provide insight into the philosophical question of whether truth may be plausibly conceived to be a thin or insubstantial notion. From a broader perspective, this research can be viewed as a test-case for the extent to which one can present, in the context of axiomatized theories, first-order approximations of a properly higher-order concept.

However, it is obviously natural to consider the strength of higher-order resources in the setting of the structures themselves, as opposed to the associated theories. Here the appropriate measure of complexity is that of definability across a uniform semantics for the higher-order notions. By way of motivation, note that while Tarski's result tells us that first-order arithmetical truth is not first-order definable in the natural numbers, it is easy to see that first-order validity is first-order definable in the natural numbers (since by the completeness theorem it aligns with provability). One topic discussed by Väänänen is a counterpoint to this in the setting of second-order logic. In particular, Väänänen proves that amongst the structures in which second-order validity is definable (in a second-order manner), none of these structures have finitely axiomatizable categorical second-order descriptions. That is, no such structure can be distinguished from the other structures by virtue of it satisfying any finite set of second-order sentences. Given that there is an obvious sense in which structures possessing such categorical descriptions are to be privileged, this result isolates an important trade-off of shifting to higher-order logics: while we are able to define truth, we are no longer able to define validity.

The focus of Welch's tutorial was an introduction to inner model theory, centered around an exposition of the set-theoretic implications of various higher-order determinacy hypotheses. To fix terminology, consider the two-player game wherein the players take turns choosing 0's and 1's, so that the result of an infinite game is an infinite sequence of 0's and 1's. Given a class of such infinite sequences, the game with respect to this class is such that the first player wins if and only if the infinite sequence associated to the game is in the class. Further, this game is said to be determined if one of the players has a winning strategy. A well-known theorem, called Borel determinacy, says that the game with respect to a class is determined if the class is not defined by quantification over infinite sequences. The theorem exposited by Welch implies that this is the best result that can be proved in set theory. It does so by showing that determinacy hypotheses with respect to classes defined by a single quantification over infinite sequences are equivalent to claims to the effect that Gödel's constructible universe differs in various specific manners from the universe of all sets.

2.2 Hierarchies: from Type Theory to Large Cardinals

The cumulative hierarchy of set-theory is of course the following sequence of sets, indexed by ordinals, wherein $P(X)$ denotes the set of all subsets of X:

$$V_0 = \emptyset, \quad V_{\alpha+1} = P(V_\alpha), \quad V_\alpha = \bigcup_{\beta<\alpha} V_\beta \text{ if } \alpha \text{ limit}.$$

One of the basic features of the ZFC set-theory axioms is that they prove that every set is an element of one of these sets. By Gödel's results, it is consistent with these axioms that this hierarchy eventually align with the thinner constructible hierarchy, wherein $\mathrm{Def}(X)$ denotes the first-order definable subsets of the set X:

$$L_0 = \emptyset, \quad L_{\alpha+1} = \mathrm{Def}(L_\alpha), \quad L_\alpha = \bigcup_{\beta<\alpha} L_\beta \text{ if } \alpha \text{ limit.}$$

These hierarchies and their initial segments have traditionally been thought significant because they provide natural models of various of the set-theoretic axioms and related systems.

For instance, in his tutorial, Ferreira described one of the classical results in this area, namely that initial segments of the constructible hierarchy align with certain "tame" elements of the first infinite level of the cumulative hierarchy. More specifically, the structure L_α for α countably infinite are natural models of predicative type theory, wherein one simply adds layers upon layers of definable sets one after the other a countable number of times. It is thus natural to ask what elements of $P(\omega) \subseteq V_{\omega+1}$ are contained in these L_α. It is a classical result, due in various guises to Kleene, Boolos, and Putnam, that a subset X of natural numbers is in L_α, where α is the least non-recursive ordinal, if and only if X has both a Σ^1_1 and Π^1_1-definition on the structure $(\omega, P(\omega))$, e.g. can be defined by one existential quantifier and by one universal quantifier over subsets of natural numbers. Thus, what this classical result tells us is that sets of natural numbers with definitions of this form are exactly the sets of natural numbers contained in a natural model of predicative type theory.

The tutorial of Incurvati and Leitgeb focused on the philosophical underpinnings of one feature of the cumulative hierarchy, namely, that there are no infinite descending sequences of sets, each is contained in its predecessor. In its usual presentation, this is a consequence of the fact that the ZFC axioms contain the axiom of foundation. In his portion tutorial, Incurvati considered potential motivations for an alternative conception of set that contains controlled violations of the axiom of foundation. One such motivation, for instance, is the idea that sets are merely the mathematical residue of certain graphs, e.g. what is obtained when the graph-theoretic structure is "forgotten." In Leitgeb's portion of the tutorial, the focus was on the idea that there may be analogues of the axiom of foundation in the setting of theories of truth, the underlying thought being that sets containing themselves might be akin to the "liar sentences" used to prove Tarski's Undefinability Theorem.

In their workshop talks, both Welch and Bagaria focused on axiomatic modes of extending the length of the cumulative hierarchy. Both of these talks develop in various ways ideas of Reinhardt, who attempted to axiomatize the scenario in which there is a non-trivial elementary embedding j from V_κ to some larger V_ζ (where κ is the least α such that $j(\alpha) > \alpha$). This scenario is of interest because it can be shown to imply the existence of rather large cardinals. Bagaria focused on variations of this called *structural reflection principles*. These principles say that if one has a definable class of structures in the signature of set theory then there is a level of cumulative hierarchy such that every structure in the class has an elementary substructure with an isomorphic copy in the class of that level. Bagaria proves that even when one allows comparatively low levels of definability (measured by alternations of blocks of quantifiers), these structural reflection principles imply the presence of the scenarios envisioned by Reinhardt, and thus imply the existence of large cardinals.

2.3 The Metamathematics of Set-Theoretic Independence

Several of the talks and tutorials focused on general metamathematical features of forcing. This is the method, invented by Cohen, for showing the independence of the continuum hypothesis and other set-theoretic statements. The very rough idea is that one shows this independence by taking a model M of set theory (or a fragment thereof) and adding an object G with "no specific features" to obtain a model $M[G]$, the so-called generic extension of M by G, which may model the negation of statements like the continuum hypothesis.

In his tutorials and workshop talk, Hamkins considered two metamathematical facets of forcing. First, he examined the propositional modal logic given by declaring one model accessible from another if it is a generic extension of the first. This turned out to be exactly the same propositional modal logic as that of directed partially ordered frames (a logic well-known to modal logicians). Second, Hamkins showed that the ostensibly second-order statement "the universe is not a generic extension of some smaller universe" is in fact first-order expressible, and he examined the independence of this first-order statement from other well-known statements of set theory.

In his tutorial talk, Bagaria exposited some elements of Woodin's argument against the vagueness of the continuum hypothesis. The basic idea behind the argument is this. The Ω-conjecture says that "truth in all generic extensions" lines up with a certain quasi-proof relation which happens to be definable in a large structure M that we get from large cardinal axioms (just like ordinary proof is definable in the standard model of arithmetic). Consider now sentences of the form: "M models p." If "M models p" is true in all generic extensions then it is actually true (since the universe is trivially a generic extension).

Suppose, for a moment, that the converse held, that is, suppose whenever "M models p" was true it was also that case that it was true in all generic extensions. Then, under the hypothesis of the Ω-conjecture, "M models p" would be definable in M, contradicting Tarski's Undefinability Theorem. Thus, we can conclude that the Ω-conjecture implies that there is statement of the form "M model p" that is true but is false in some generic extension. It turns out that this behavior persists, so that "M models p" whilst true can always be forced to be true and can always be forced to be false (over models with the appropriate large cardinal assumptions). Thus, the Ω-conjecture implies the existence of a true statement such that both it and its negation are false in some forcing extension. Thus, the moral is that being false in a forcing extension is no reason to reject the truth of a set-theoretic statement, and so one ought to refrain from so rejecting the truth of the continuum hypothesis.

Finally, the workshop talk of Mathias focused on the extent to which forcing could be developed in axiom systems weaker than ZFC set-theory. In particular, the traditional development of forcing employs heavily the replacement schema, which is unavailable in set theories that have models like $V_{\omega+\omega}$ or L_α, where α is the least non-recursive ordinal. Thus Mathias focused on minimal additions to the natural axioms for these structures which would allow one to do forcing over them. His intended applications are to weak variants of determinacy, e.g. he seeks to show the independence of weak variants of determinacy over weak subsystems of set-theory by appeal to this rarified notion of forcing.

3 Assessment of the Results and Impact

There are a number of different dimensions along which one might attempt to gauge the significance or importance of a meeting. On a sociological level, this meeting was quite successful at bringing together the set theory and philosophy of math communities in Europe, and so was significant in this respect. It was also important in that it filled a pedagogical need for both communities, as the tutorial components provided a very concrete introduction to an adjacent discipline for students. Some admittedly non-exhaustive discussions with several student participants seemed to indicate that they found few similar avenues for such interdisciplinary studies at their home institutions.

Another readily visible indicator of the significance of the summer school is the fact that a special journal issue of the *Notre Dame Journal of Formal Logic* is being arranged to feature papers from the workshop component of the summer school. These papers will be individually and externally refereed by the usual standards of scholarly journals. It is presently anticipated that this issue will be forthcoming sometime in 2012, with circa 6-10 article submissions.

A third and final measure of potential impact is the large number of open questions that were articulated in the course of the summer school. Many of these questions are comparatively basic questions about the relevant concepts, as is indicated by the brevity with which many can be posed. This suggests to us that many of the topics treated here are new and that there is much room for healthy growth in and around these questions. For the sake of definiteness, let us explicitly reiterate five of these questions (as well as the speakers who posed them):

1. Can large cardinals below supercompact be characterized in terms of structural reflection? (Bagaria)

2. Recalling that a ground is an inner model of which the universe is a forcing extension, and that a bedrock is ground that is minimal among all grounds, is the bedrock unique when it exists? (Hamkins)

3. Are there any global degrees of interpretability that are first-order definable in terms of the double degree structure [i.e. with both local and global interpretability] other than the minimal global degree and the maximal global degree in the minimal local degree? (Visser)

4. Can the consistency of Heck's predicative set theory be proven finitistically? (Ferreira)

5. What is the consistency strength of the Completeness Theorem for Boolean valued second order logic consistent? (Väänänen)

4 Speaker List

1. Aldo Antonelli (UC Davis),
   email: antonelli@ucdavis.edu, website: link
2. Joan Bagaria (Barcelona),
   email: joan.bagaria@icrea.cat, website: link
3. Fernando Ferreira (Lisbon),
   email: fjferreira@fc.ul.pt, website: link
4. Joel David Hamkins (CUNY, NYU),
   email: jhamkins@gc.cuny.edu, website: link
5. Leon Horsten (Bristol),
   email: leon.horsten@bristol.ac.uk, website: link
6. Luca Incurvati (Cambridge),
   email: li216@cam.ac.uk, website: link
7. Hannes Leitgeb (Munich, Center for Mathematical Philosophy),
   email: Hannes.Leitgeb@lmu.de, website: link
8. Adrian Mathias (Université de la Réunion),
   email: A.R.D.Mathias@dpmms.cam.ac.uk, website: link
9. Jouko Väänänen (Amsterdam, Helsinki),
   email: jouko.vaananen@helsinki.fi, website: link
10. Albert Visser (Utrecht),
    email: Albert.Visser@phil.uu.nl, website: link
11. Philip Welch (Bristol),
    email: P.Welch@bristol.ac.uk, website: link

5 Full Participant List

1. Aldo Antonelli (UC Davis),
email: antonelli@ucdavis.edu, website: link

2. Marianna Antonutti (Bristol),
email: plmam@bristol.ac.uk, website: link

3. Carolin Antos-Kuby (Kurt Gödel Research Center, Vienna),
email: carolin.antos-kuby@univie.ac.at, website: link

4. Giorgio Audrito (Torino),
email: giorgio.audrito@gmail.com

5. Joan Bagaria (Barcelona),
email: joan.bagaria@icrea.cat, website: link

6. Neil Barton (UCL),
email: bartonna@gmail.com, website: link

7. Robert Black (Nottingham),
email: mongre@gmx.de, website: link

8. Joshua Black (Oxford),
email: jdb106@uclive.ac.nz

9. Francesca Boccuni (University Vita-Salute San Raffaele, Milan),
email: francesca.boccuni@tiscali.it, website: link

10. Jason Costanzo (St. John's University),
email: jasoncostanzo@hotmail.com, website: link

11. Raj Dahya (Melbourne),
email: r.dahya@pgrad.unimelb.edu.au, website: link

12. Michael De (Utrecht),
email: mikejde@gmail.com, website: link

13. Walter Dean (Warwick),
email: w.h.dean@warwick.ac.uk, website: link

14. Michael Detlefsen (Notre Dame, ANR),
email: Michael.Detlefsen.1@nd.edu, website: link

15. Harry Deutsch (Illinois State University),
email: hdeutsch@ilstu.edu, website: link

16. Benedict Eastaugh (Bristol),
email: be2923@bristol.ac.uk, website: link

17. Pedro Falcao (São Paulo),
email: satisfac@gmail.com

18. Fernando Ferreira (Lisbon),
email: fjferreira@fc.ul.pt, website: link

19. José Ferreirs (Sevilla),
email: josef@us.es, website: link

20. Salvatore Florio (Birkbeck),
email: s.florio@bbk.ac.uk, website: link

21. Kentaro Fujimoto (Oxford),
email: fujimotokentaro@gmail.com, website: link

22. Luz Mara Garca vila (Barcelona),
email: luzsabinaw@gmail.com, website: link

23. Peter Gibson (Birkbeck),
email: petermagibson@gmail.com, website: link

24. CS Gifford (Bristol),
email: Chris.Gifford@bristol.ac.uk, website: link

25. Zaln Gyenis (CEU (Budapest)),
email: gyz@renyi.hu, website: link

26. Simon Hewitt (Birkbeck),
email: sublimeobject@gmail.com, website: link

27. Ole Hjortland (Munich, Center for Mathematical Philosophy),
email: olethhjortland@gmail.com, website: link

28. Kate Hodesdon (Bristol),
email: kate@hodesdon.com, website: link

29. Leon Horsten (Bristol),
email: leon.horsten@bristol.ac.uk, website: link

30. Luca Incurvati (Cambridge),
email: li216@cam.ac.uk, website: link

31. Daniel Isaacson (Oxford),
email: daniel.isaacson@philosophy.ox.ac.uk, website: link

32. Joel David Hamkins (CUNY, NYU),
    email: jhamkins@gc.cuny.edu, website: link

33. Roger Jones (Independent Scholar),
    email: rbj@rbjones.com, website: link

34. Wilfried Keller (Göttingen),
    email: Wilfried.Keller@creativehq.de, website: link

35. Juliette Kennedy (Helsinki),
    email: jcarakenned@gmail.com, website: link

36. Jeff Ketland (Munich, Center for Mathematical Philosophy),
    email: Jeffrey.Ketland@lrz.uni-muenchen.de, website: link

37. Johannes Korbmacher (Münster),
    email: jkorbmacher@googlemail.com, website: link

38. Kevin Kuhl (Toronto),
    email: kevin.kuhl@utoronto.ca, website: link

39. Ramana Kumar (Cambridge),
    email: rk436@cam.ac.uk, website: link

40. Elaine Landry (UC Davis),
    email: emlandry@ucdavis.edu, website: link

41. Graham Leach-Krouse (Notre Dame),
    email: gleachkr@nd.edu, website: link

42. Hannes Leitgeb (Munich, Center for Mathematical Philosophy),
    email: Hannes.Leitgeb@lmu.de, website: link

43. Julio Lemos (So Paulo),
    email: old.mores@gmail.com, website: link

44. Guanhao Li (Aberdeen),
    email: g.li.09@aberdeen.ac.uk

45. Liam Bright (LSE),
    email: liamkbright@gmail.com

46. Øystein Linnebo (Birkbeck),
    email: o.linnebo@bbk.ac.uk, website: link

47. Margaret MacDougall (Edinburgh),
    email: Margaret.MacDougall@ed.ac.uk, website: link

48. Alex Malpass (Bristol),
email: am6605@bristol.ac.uk, website: link

49. Kate Manion (King's),
email: kate.manion@blueyonder.co.uk, website: link

50. Laura Mrquez (Bristol),
email: l.marquez.07@bristol.ac.uk, website: link

51. Adrian Mathias (Université de la Réunion),
email: A.R.D.Mathias@dpmms.cam.ac.uk, website: link

52. Toby Meadows (Arche (St. Andrews)),
email: toby.meadows@gmail.com, website: link

53. Julien Murzi (Munich, Center for Mathematical Philosophy),
email: j.murzi@gmail.com, website: link

54. Carlo Nicolai (Oxford),
email: carlo.nicolai@some.ox.ac.uk, website: link

55. Jonathan Payne (Sheffield),
email: jdapayne@gmail.com, website: link

56. Sebastian Petzolt (Oxford),
email: sebastian.petzolt@st-hildas.ox.ac.uk, website: link

57. Samantha Pollock (Bristol),
email: sp8777@bristol.ac.uk

58. Michael Potter (Cambridge),
email: mdp10@cam.ac.uk, website: link

59. Grant Reaber (Aberdeen (NIP)),
email: grant.reaber@gmail.com, website: link

60. Sam Roberts (Birkbeck),
email: srober21@mail.bbk.ac.uk, website: link

61. Joan Rosell Moya (Independent Scholar),
email: joanrosello@gmail.com, website: link

62. Paul Ross (Bristol),
email: pauladr2@googlemail.com

63. Ramez L Sami (Paris VII),
email: sami@logique.jussieu.fr, website: link

64. Gonalo Santos (Barcelona),
    email: goncalo.b.santos@gmail.com, website: link

65. Gregor Schneider (Munich, Center for Mathematical Philosophy),
    email: gfs@gfschneider.de, website: link

66. Jönne Speck (Birkbeck),
    email: jonne@runbox.com, website: link

67. Marta Sznajder (Munich, Center for Mathematical Philosophy),
    email: mszn@gazeta.pl,

68. Fenner Tanswell (Amsterdam),
    email: Fenner.Tanswell@student.uva.nl, website: link

69. Paul Taylor (Independent Scholar),
    email: pt11@paultaylor.eu, website: link

70. Claudio Ternullo (Liverpool),
    email: C.Ternullo@liverpool.ac.uk, website: link

71. Giulia Terzian (Bristol),
    email: giulia.terzian@gmail.com, website: link

72. AR Thomas-Bolduc (Bristol),
    email: at6131@bristol.ac.uk

73. Lauri Tuomi (Paris VII),
    email: tuomi@logique.jussieu.fr, website: link

74. Jouko Väänänen (Amsterdam, Helsinki),
    email: jouko.vaananen@helsinki.fi, website: link

75. Giorgio Venturi (Paris VII),
    email: giorgio.venturi@sns.it, website: link

76. Albert Visser (Utrecht),
    email: Albert.Visser@phil.uu.nl, website: link

77. Jason Adam Voss (Eastern Michigan),
    email: jason.adam.voss@gmail.com

78. Sean Walsh (Birkbeck),
    email: swalsh108@gmail.com, website: link

79. Philip Welch (Bristol),
    email: P.Welch@bristol.ac.uk, website: link

80. John Wigglesworth (CUNY),
email: jmwigglesworth@gmail.com, website: link

81. John Woods (Princeton),
email: jewoods@princeton.edu, website: link

82. Leszek Wronski (Jagiellonian (Kraków)),
email: leszek.wronski@uj.edu.pl

6 Full Program

6.1 Time-Tables, Venue, Dinners, Map

Time-Table for Mini-Courses (August 1-4)

	Monday	Tuesday	Wednesday	Thursday
10:00-10:50	Väänänen	Ferreira	Visser	Horsten
11:00-11:50	Welch	Bagaria	Hamkins	Visser
12:00-12:50	Ferreira	Väänänen	Horsten	Incurvati-Leitgeb
3:00-3:50	Hamkins	Welch	Bagaria	Hamkins
4:00-4:50	Visser	Horsten	Väänänen	Ferreira
5:00-5:50	Incurvati-Leitgeb	Incurvati-Leitgeb	Welch	Bagaria

Time-Table for Conference (August 5-6)

	Friday	Saturday
10-11:15	Mathias	Bagaria
11:30-12:45	Väänänen	Horsten
2:15-3:30	Welch	Visser
3:45-5:00	Antonelli	Incurvati
5:15-6:30	Ferreira	Hamkins

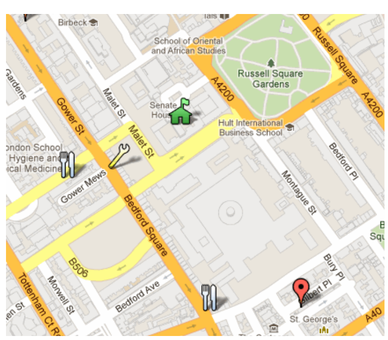
Venue: Room ST274/275 in the Institute of Philosophy, Stewart House, 32 Russell Square. (The "home" image in the below map).

First Conference Dinner (Monday, August 1, 7:00pm). Tas Restaurant, 22 Bloomsbury Street. (The lower "fork and knives" image in the below map).

Second Conference Dinner (Friday, August 5, 7:00pm). Pizza Paradiso, 35 Store Street. (The upper "fork and knives" image in the below map).

Note on Dinners: Reservations have been made for speakers. If you are not in this group, please contact Sean Walsh to inquire about availability and costs.

Map: This is the conference map. Clicking on it will bring you to a full googlemap:



6.2 Abstracts for Mini-Courses (August 1-4)

6.2.1 Joan Bagaria (Barcelona),
An introduction to Omega-logic

Abstract: In 1990, W. Hugh Woodin introduced Omega-logic as an approach to truth in the universe of all sets inspired by recent work on large cardinals and determinacy. In Omega-logic statements are valid if, roughly, they hold in every forcing extension of the universe, or its sufficiently rich initial segments. Thus, Omega-logic is the logic of generic absoluteness. In this short course we will start with a brief introduction to forcing, large cardinals, and generic absoluteness. We will then describe the basic features of Omega-logic, leading to the formulation of the Omega-conjecture and a discussion of its relevance in current set-theoretic research. *Suggested Background Reading*: [BCL06], [Jec03], [Kan03], [Kun80], [Woo99], [Woo01].

6.2.2 Fernando Ferreira (Lisbon),
An Overview of Predicativity

Abstract: The classical discussion on predicativity: Poincaré and Russell. The logicist project. Ramification and reducibility. The consistency of Frege's predicative system. The impredicativity of induction. What can be done in a strict predicative system? Finite reducibility. Predicativity given the natural numbers. Weyl's approach. Kreisel's proposals. The hyperarithmetic sets. Predicative provability and the Feferman-Schütte proof-theoretical analysis. Predicative mathematics. A working hypothesis concerning the indispensability arguments. The non-realistic cast of predicativism. A predicative logic? Reverse mathematics and the limits of predicativity. Impredicativity and realism. *Suggested Background Readings*: [Fef05], [Hec96] (on a first reading, perhaps skip section 3), [Par02], [Sim02].

6.2.3 Luca Incurvati (Cambridge) and Hannes Leitgeb (Munich),
Groundedness in Set Theory and Semantics

Abstract: This three-lectures course will focus on groundedness in set theory and semantics. Topics to be discussed on the set-theoretic side will be: the iterative conception of sets; differences and connections between various groundedness assumptions for sets; the issue of the justification of these assumptions and of assumptions violating groundedness; the connection, if any, between groundedness and the logic of set theory. And on the semantic side: grounded instances of the Tarskian T-scheme; semantic dependency; dependency vs. membership; and a note on grounded abstraction. *Suggested Background Readings*: [Inc11], [Lei05], [Lin08], [Rie00].

6.2.4 Joel Hamkins (CUNY, NYU),

A Tutorial on Set-Theoretic Geology

Abstract: The technique of forcing in set theory is customarily thought of as a method for constructing outer as opposed to inner models of set theory; one starts in a ground model V and considers the possible forcing extensions V[G] of it. A simple switch in perspective, however, allows us to use forcing to describe inner models, by considering how a given universe V may itself have arisen by forcing. This change in viewpoint leads to the topic of set-theoretic geology, aiming to investigate the structure and properties of the ground models of the universe. In this tutorial, I shall present some of the most interesting initial results in the topic, along with an abundance of open questions, many of which concern fundamental issues.

A ground of the universe V is an inner model W of ZFC over which the universe V=W[G] is a forcing extension. The model V satisfies the Ground Axiom of there are no such W properly contained in V. The model W is a bedrock of V if it is a ground of V and satisfies the Ground Axiom. The mantle of V is the intersection of all grounds of V, and the generic mantle is the intersection of all grounds of all set-forcing extensions. The generic HOD, written gHOD, is the intersection of all HODs of all set-forcing extensions. The generic HOD is always a model of ZFC, and the generic mantle is always a model of ZF. Every model of ZFC is the mantle and generic mantle of another model of ZFC, and this can be proved while also controlling the HOD of the final model, as well as the generic HOD. Iteratively taking the mantle penetrates down through the inner mantles to the outer core, what remains when all outer layers of forcing have been stripped away. Many fundamental questions remain open. *Suggested Background Readings*: [Jec03], [Kun80], [Bel85], [Ham09]

6.2.5 Leon Horsten (Bristol),

Truth and Paradox

This mini-course will be on the concept of truth and the semantic paradoxes associated with it. The emphasis will be on axiomatic theories of truth that are motivated by models of truth. We will also explore the connection between axiomatic truth theories and deflationism about truth. The structure of the course will be roughly as follows: (1) Introduction: Truth and the Liar, (2) Definition, Models, Axioms, (3) Typed Disquotational Theories, (4) Typed Compositional Theories, (5) Type-free Disquotational Theories, (6) Type-free Compositional Theories, (7) The Revision Theory of Truth. *Suggested Background Reading*: [Hal10], [Hor11]

6.2.6 Jouko Väänänen (Helsinki, Amsterdam),

Model Theory of Second Order Logic

I will give an introduction to the semantics of second order logic, both in the sense of full second order logic and in the sense of Henkin semantics. The main issue is the categoricity of various second order theories and the related issue of second order characterizations of important mathematical structures. *Suggested Background Readings*: [Vää01], [Sha91], [Jan05], [Vää11].

6.2.7 Albert Visser (Utrecht),
Interpretations

When is a theory stronger than another theory? When are two theories the same? How can we reduce the undecidability of one theory to the undecidability of another? Interpretations play an important role in the answers to these questions. In this course we aim to give a substantial introduction to the subject of interpretations.

> Lecture 1: Interpretations Introduced. In this lecture we provide the definition of what interpretations are. Specifically, we discuss subjects like domain relativization, non-preservation of identity, parameters, many-dimensional interpretations and piecewise interpretations. We consider categories of interpretations. These will be used to define several notions of sameness of theories. Finally we consider reduction relations based on interpretations like global interpretability, model interpretability and local interpretability.
>
> Lecture 2: What is a Sequential Theory. Sequential theories are in essence theories with coding. Sequential theories have an extremely simple definition from which rich properties can be extracted. We sketch the impressive bootstrap —a sequence of interpretations— that provides these rich properties. We give the Gödel-Hilbert-Bernays-Wang-Henkin-Feferman theorem that allows us to construct the interpretation of a theory from a consistency statement for that theory. We discuss two important consequences of the GHBWHF-theorem: the Orey-Hájek and the Friedman characterizations of interpretability.
>
> Lecture 3: Capita Selecta on Degrees of Interpretability. In this lecture we consider different degree structures for interpretability and we prove some properties of these structures. For example, the various degree structures are distributive lattices, many of them are dense, etcetera. The focus of the lecture will be not so much on providing a comprehensive list of results but on illustrating the nice proof methods. We end the lecture with a some questions: there is still a lot of work to do here!

Suggested Background Readings: [HP98], [Lin03], [Vis06]

6.2.8 Philip Welch (Bristol),
Inner model theory, Large Cardinals and Determinacy

In these three lectures we aim to introduce some of the fundamental features of modern set theory: a) that there is a potential spectrum of *inner models* of the ZFC axioms, generalising Gödel's L; b) that the existence of embedding properties of such a model as L has a profound effect on how we view the universe V of all sets; c) that such embedding properties of inner models, and in particular of V generate a hierarchy of large cardinal properties, or "strong axioms of infinity"; d) that such large cardinal properties although ostensibly about sets in remote regions of the iterative hierarchy, in fact affect how we view the real continuum, and thus $V_{\omega+1}$, in a substantiative way; e) how these phenomena are all mediated through the concept of *determinacy* of infinite integers games.

Lecture I. Inner Model theory and Covering Theorems

Lecture II. Large Cardinals and Tree Representations of sets of reals.

Lecture III. Determinacy.

The aim is to give an introductory account that nevertheless attempts to give a sufficient overview of the interrelationships by focussing on some particular examples of these constructions. It should be accessible to someone familiar with a development of some set theory from the ZFC axioms, and the background reading. We shall assume knowledge of:

a) the *ultrapower construction* of a first order structure $(M, \in)$ using an ultrafilter;

b) the standard Baire space topology on $\mathbb{N}^{\mathbb{N}}$ (or ω^ω), and its relation to sets definable in SO number theory. Helpful: representation of Π^1_1-sets as trees.

c) Helpful will also be: if the listener has seen a development of the Gödel constructible sets as a hierarchy L_α, and the Condensation Lemma/GCH argument there (the details of these constructions arguments will be referred to but not given in detail).

Preparatory Reading. Background reading for the whole enterprise is [Jen95] (a very approachable and succinct general lecture given by Jensen). The first parts of [HKP11] givs a very readable introduction to the notion of elementary embeddings between inner models, in particular considerations of when such are not definable.

For these lectures: a) see Jech [Jec03] Sect. 12.. We shall handwave over *iterated ultrapower* constructions ([Jec03] Sect. 19 or Kanamori [Kan03], Sect 19). However the notion of *direct limit* of structures will be mentioned. Koepke, [Koe02] (up to 4.4), gives an account of inner models considered roughly as a category with elementary embeddings as morphisms with diect limits. For b) Moschovakis [Mos80] is the standard reference, but sufficient detail is also in Jech Sect. 25. For c) again Devlin [Dev84] is the usual full reference, but any account of L and GCH will do, for example Jech, or [DS96].

For intermediate reading: the later parts of [Koe02] places its emphasis on embedding representations of sets of reals.

For subsequent, rather advanced reading: the Martin-Steel theorem appears in [MS89], but an account using embedding normal forms is in [Koe98]. In the *Handbook of Set Theory* [FK10]. consult the chapter on the Covering Lemma by Mitchell; obtaining models with large cardinals from determinacy is in general difficult, see the *Handbook* article by Koellner and Woodin here; Neeman has a chapter here on determinacy in $L(\mathbb{R})$; there are two chapters on fine structure of inner models by Schindler and Zeman, and this author, but for those seriously interested in learning the interaction of inner models with covering and the construction of core models, then they should study [Zem02].

6.3 Abstracts for Conference (August 5-6)

6.3.1 Aldo Antonelli (UC Davis),
The Abstraction Mystique

Abstract: The logical status of abstraction principles, and especially Humes Principle, has been long debated, but the best currently available tool for explicating a notions logical character – the notion of permutation invariance introduced by Tarski in 1966 – has not received a lot of attention in this debate. This talk aims to fill this gap. After characterizing abstraction principles as particular classifiers, i.e., mappings from the subsets of a domain into that domain and exploring some of their properties, the paper introduces several distinct notions of permutation invariance for such principles, assessing the philosophical significance of each.

6.3.2 Joan Bagaria (Barcelona),
Second-order-indescribable cardinals and topologies on ordinals

Abstract: I will present some preliminary results connecting second-order-indescribable cardinals with natural topologies on ordinals of uncountable cofinality. The general goal is to give a characterization of (some) large cardinals in terms of the existence of certain natural topologies on them.

6.3.3 Fernando Ferreira (Lisbon),
A note on Spector's proof of consistency of analysis

Abstract: In 1962, Clifford Spector gave a consistency proof of analysis using so-called bar-recursors. His paper extends an interpretation of arithmetic given by Kurt Gdel in 1958. Spector's proof relies crucially on the interpretation of the so-called (numerical) double negation shift principle. We explain the role of this principle in interpreting full numerical comprehension. Spector's interpretation is ad hoc. On the other hand, William Howard gave in 1968 a very natural interpretation of bar-induction by bar-recursion. We show that, within the framework of Gdel's interpretation, (numerical) double negation shift is a consequence of bar-induction. Actually, this result can be seen as particular case of Howard's work, but our (simple) derivation may be of independent interest.

6.3.4 Luca Incurvati (Cambridge),
A conception of set-theoretical truth

Abstract: Daniel Isaacson has argued that we have a conception of arithmetical truth for which Peano Arithmetic is sound and complete. Is there an analogue of Isaacson's Thesis for set theory? In previous work, I have discussed Leon Horsten's proposal for providing one. In this talk, I will try and say something more positive on this issue by formulating a conception of set-theoretical truth and assessing the prospects for finding a system sound and complete with respect to it. Along the way, I will also say something on Isaacson's own attempt to provide an analogue of his Thesis for set theory.

6.3.5 Joel Hamkins (CUNY, NYU), *The modal logic of forcing*

Abstract: What are the most general principles relating forceability and truth? As in Solovay's celebrated analysis of provability, this question and its answer are naturally formulated in modal logic. Specifically, we define that a set theoretic assertion phi is *forceable* or *possible* if it holds in some forcing extension, and *necessary* if it holds in all forcing extensions. Under this forcing interpretation, the provably valid principles of forcing are exactly those in the modal theory known as S4.2. In this talk, I shall discuss this result and recent advances in this area. This is joint work with Benedikt Loewe.

6.3.6 Leon Horsten (Bristol), *TBA*

Abstract: TBA.

6.3.7 A.R.D. Mathias (Université de la Réunion), *The separation of powers: weak subsystems of ZF in the service of philosophy*

Abstract: The talk will begin with a "slow boot" of set theory, passing through the system MW, which corrects the chief problem with Devlins book *Constructibility*, and reaching GJ, the set theory whose transitive models are those closed under the functions called "basic" by Gandy and "rudimentary" by Jensen. Then the notions of rudimentary recursion and provident set will be introduced, and the set theory PROVI presented, together with a counterexample and a technical improvement due to Nathan Bowler. That PROVI suffices for a development of the general theory of set forcing will be sketched, and the barest outline given of a proof that a set-generic extension of a provident set is provident. Then the talk will turn to examining the power set axiom and two set theories, proposed by Zermelo and Mac Lane, in which it plays a major role. The usual machinery of forcing can go wrong when applied to transitive models of these theories; but the situation is saved by moving either to the provident closure or, more lavishly, to the lune of such models. The relationship of Z and MAC to the familiar Kripke-Platek system of Σ_1-recursion will be discussed, as will Harvey Friedmans system $KP^{\mathcal{P}}$. Finally, implications of these technical results for the philosophy of mathematics will be suggested.

6.3.8 Jouko Väänänen (Helsinki, Amsterdam), *Set theory or higher order logic?*

Abstract: The question, whether higher (or even just second) order logic is a better foundation for mathematics than set theory, is addressed. The main difference between higher order logic and set theory is that set theory builds up a transfinite cumulative hierarchy while higher order logic stays within a specific number of applications of the power sets, maybe

just one, as in second order logic. It is argued that in many ways this difference is illusory. More importantly, it is argued that the often stated difference, that higher order logic has categorical characterizations of relevant mathematical structures, while set theory has non-standard models, amounts to no difference at all. HIgher order logic and set theory permit quite similar categoricity results on one hand, and similar non-standard models on the other hand. We also give some results which seem to suggest that there are serious problems in trying to use second order logic to understand second order truth without recourse to set theory.

6.3.9 Albert Visser (Utrecht),
Coordinate Free Versions of the Second Incompleteness Theorem (revisited)

Abstract: Is it possible to give a coordinate free formulation of the Second Incompleteness Theorem? We discuss two possible approaches to this question. Both approaches are inspired by FOM contributions of Harvey Friedman. In the first approach, we introduce functors SEQ and PC on theories, such that, for a finitely axiomatized theory A, SEQ(A) is mutually interpretable with the cutfree consistency of A, and PC(A) is mutually interpretable with the ordinary consistency of A. It follows that both cutfreecon(A) and con(A) are uniquely determined modulo EA-provable equivalence. The approach can be generalized to RE theories, but things are somewhat delicate in that case since we have to deal with 'intensional' phenomena. In the second approach, we do not try to characterize consistency statements but we explicate what it is for one RE theory to have consistency strength over another RE theory. We give an example of a sentence that has consistency strength over a theory but is weaker than an ordinary consistency statement.

6.3.10 Philip Welch (Bristol),
Strong Reflection Principles

Abstract: It is said that reflection principles as usually considered can only establish large cradinal properties consistent with V=L. We reflect on this situation and propose a more fruitful reflection principle implying inter alia projective determinacy and arbitrarily large Woodin cardinals - reasons for the latter include that it ensures the absolutness of the theory of the reals (thus including PD) as well as forming a working basis for many of Woodin's results on Omega-Logic.

References

[BCL06] Joan Bagaria, Neus Castells, and Paul Larson. An Ω-Logic Primer. In Set Theory, Trends in Mathematics, pages 1–28. Birkhäuser, Basel, 2006.

[Bel85] J. L. Bell. Boolean-Valued Models and Independence Proofs in Set Theory, volume 12 of Oxford Logic Guides. Clarendon, New York, second edition, 1985.

[Dev84] Keith J. Devlin. Constructibility. Perspectives in Mathematical Logice. Springer-Verlag, Berlin, 1984.

[DS96] F. R. Drake and D. Singh. Intermediate Set Theory. John Wiley & Sons Ltd., Chichester, 1996.

[Fef05] Solomon Feferman. Predicativity. In Stewart Shapiro, editor, The Oxford Handbook of Philosophy of Mathematics and Logic, pages 590–624. Oxford University Press, Oxford, 2005.

[FK10] Matthew Foreman and Akihiro Kanamori. Handbook of Set Theory. Springer, Berlin, 2010.

[Hal10] Volker Halbach. Axiomatic Theories of Truth. Oxford University Press, Oxford, 2010.

[Ham09] Joel David Hamkins. Some Second Order Set Theory. In Logic and Its Applications, volume 5378 of Lecture Notes in Computer Science, pages 36–50. Springer, Berlin, 2009.

[Hec96] Richard G. Heck, Jr. The Consistency of Predicative Fragments of Frege's Grundgesetze der Arithmetik. History and Philosophy of Logic, 17(4):209–220, 1996.

[HKP11] Joel David Hamkins, Greg Kirmayer, and Norman Lewis Perlmutter. Generalizations of the Kunen Inconsistency. Unpublished. Dated June 10, 2011.

[Hor11] Leon Horsten. The Tarskian Turn. Deflationism and Axiomatic Truth. MIT Press, Cambridge, 2011.

[HP98] Petr Hájek and Pavel Pudlák. Chapter III: Self-Reference. In Metamathematics of First-Order Arithmetic, Perspectives in Mathematical Logic. Springer, Berlin, 1998.

[Inc11] Luca Incurvati. How to be a Minimalist about Sets. Philosophical Studies, TBA(TBA):TBA, 2011.

[Jan05] Ignacio Jané. Higher-Order Logic Reconsidered. In Stewart Shapiro, editor, The Oxford Handbook of Philosophy of Mathematics and Logic, pages 781–810. Oxford University Press, Oxford, 2005.

[Jec03] Thomas Jech. Set Theory. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The Third Millennium Edition.

[Jen95] Ronald Jensen. Inner Models and Large Cardinals. Bulletin of Symbolic Logic, 1(4):393–407, 1995.

[Kan03] Akihiro Kanamori. The Higher Infinite. Springer, Berlin, second edition, 2003.

[Koe98] Peter Koepke. Extenders, Embedding Normal Forms, and the Martin-Steel-Theorem. Journal of Symbolic Logic, 63(3):1137–1176, 1998.

[Koe02] Peter Koepke. The Category of Inner Models. Synthese, 133(1-2):275–303, 2002.

[Kun80] Kenneth Kunen. Set Theory, volume 102 of Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1980.

[Lei05] Hannes Leitgeb. What Truth Depends On. Journal of Philosophical Logic, 34:155–192, 2005.

[Lin03] Per Lindström. Chapter 6: Interpretability. In Aspects of Incompleteness, volume 10 of Lecture Notes in Logic. Association for Symbolic Logic, Urbana, IL, second edition, 2003.

[Lin08] Øystein Linnebo. Structuralism and the Notion of Dependence. Philosophical Quarterly, 58:59–79, 2008.

[Mos80] Yiannis N. Moschovakis. Descriptive Set Theory, volume 100 of Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1980.

[MS89] Donald A. Martin and John R. Steel. A proof of projective determinacy. Journal of the American Mathematical Society, 2(1):71–125, 1989.

[Par02] Charles Parsons. Realism and the Debate on Impredicativity, 1917-1944. In Reflections on the Foundations of Mathematics, volume 15 of Lecture Notes in Logic, pages 372–389. Associaton of Symbolic Logic, Urbana, 2002.

[Rie00] Adam Rieger. An Argument for Finsler-Aczel Set Theory. Mind, 109(434):241–253, 2000.

[Sha91] Stewart Shapiro. Chapters 3-4: Theory & Meta-Theory. In Foundations without Foundationalism: A Case for Second-Order Logic, volume 17 of Oxford Logic Guides, pages 61–96. The Clarendon Press, New York, 1991.

[Sim02] Stephen G. Simpson. Predicativity: The Outer Limits. In Reflections on the Foundations of Mathematics, volume 15 of Lecture Notes in Logic, pages 130–136. Association for Symbolic Logic, Urbana, 2002.

[Vää01] Jouko Väänänen. Second-Order Logic and Foundations of Mathematics. The Bulletin of Symbolic Logic, 7(4):504–520, 2001.

[Vää11] Jouko Väänänen. Second Order Logic or Set Theory? Unpublished. Available here, 2011.

[Vis06] Albert Visser. Categories of Theories and Interpretations. In Logic in Tehran, volume 26 of Lecture Notes in Logic, pages 284–341. Association for Symbolic Logic, La Jolla, 2006. Edited by Ali Enayat, Iraj Kalantari and Mojtaba Moniri.

[Woo99] W. Hugh Woodin. The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal, volume 1 of de Gruyter Series in Logic and its Applications. de Gruyter, Berlin, 1999.

[Woo01] W. Hugh Woodin. The Continuum Hypothesis. I. Notices of the American Mathematical Society, 48(6):567–576, 2001.

[Zem02] Martin Zeman. Inner Models and Large Cardinals, volume 5 of de Gruyter Series in Logic and its Applications. Walter de Gruyter & Co., Berlin, 2002.