Scientific Report:
Conference on 2D statistical physics

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Place. Les Diablerets, SWITZERLAND
Organizer. Stanislav Smirnov

1 Scientific summary

The emergence of the renormalization group formalism and later of Conformal Field Theory led to a deep physical and geometrical understanding of planar lattice models, albeit mostly non-rigorous. In 1994, Pouilot, Langlands and Saint-Aubin discussed numerics on the scaling limit of crossing probabilities for critical percolation. The same year, Cardy predicted an explicit formula for this scaling limit. Following these two influential works, mathematicians got interested in the study of planar statistical physics from a conformal invariance point of view. As a consequence, the mathematical understanding of these models, and in particular of their continuum scaling limits at criticality, progressed greatly in the last few years.

Two breakthroughs appeared around fifteen years ago. First, the introduction of Schramm-Loewner Evolutions provided a mathematical framework to study models such as percolation, Ising, loop-erased random walks, Uniform Spanning Trees and dimers. This development was combined with the use of preholomorphic observables to show rigorously conformal invariance of lattice models, see e.g. the works of Kenyon, Cardy and Smirnov. These mathematical works created an explosion of papers, and new connections between probability and Conformal Field Theory arised.

The meeting focused on three main subjects (which are obviously very much related). The first object of interest was the conformal invariance of lattice models such as Ising models, percolation and random-cluster models. The standard strategy in order to understand the critical behavior of these models is to exhibit preholomorphic observables. The works of Chelkak, Izyurov, Poinsaing and Smirnov point in this direction. Another important component of this study is the development of a theory of crossing probabilities (for instance the Russo-Seymour-Welsh theory for Ising and random-cluster models). Chelkak, Duminil-Copin and Beffara's researchs deal with these issues. Special emphasis was also put on robustness of existing proofs of conformal invariance. Indeed, while the scaling limit of lattice models is predicted to be universal, most of the existing proofs are restricted to special lattices. It is therefore important to investigate their generality. The works of Manolescu and Cimasoni represent recent breakthroughs in the understanding of universality in percolation and Ising respectively.

Another major branch of today's planar statistical physics was at the center of this meeting, namely the theory of random planar maps. This theory has been developed over the last years in part motivated by the theory of two-dimensional quantum gravity. Very recently, Le Gall and Miermont independently showed that a large class of random planar maps admits a continuum limit, a compact random surface called the Brownian map. The Brownian map, when properly embedded in the sphere, is conjectured to be

related to Liouville Quantum Gravity. Sheffield and Duplantier progressed in the mathematical understanding of the former object by proving a weak form of the KPZ relation (this offers a striking example of the fecundity of collaborations between mathematicians and physicists). Their work on the KPZ relation was discussed during the workshop with other specialists of random planar maps such as Le Gall, Miermont, Curien and Benjamini.

Last but not least, SLE and related objects like CLE and GFF are now omnipresent in planar statistical physics. Their study is therefore crucial for the development of the field. Furthermore, they constitute very interesting objects on their own. Several talks were devoted to these objects (including the works of Garban, Miller, Sheffield, Watson, Wilson, Wu).

In conclusion, this workshop, which was held in the continuity of the Ascona meeting of 2010, provided the perfect framework for the aforementioned participants to interact and to try to progress towards a better understanding of conformal invariance of lattice models. It brought together probabilists and physicists. In the afternoon, two hours were devoted to thematic mathematical discussions. Beside senior researchers, students were also given the opportunity to give talks and participate to the mathematical discussions. This conference gave the opportunity to discuss progress achieved during the last three years (since the Ascona meeting) as well as the future challenges of the field.

2 Scientific content and discussions

The workshop was composed of talks of 55 minutes, together with two mini-courses of respectively three and four talks. The afternoon were devoted to discussions.

2.1 Mini-courses

Miermont presented a mini-course of three hours entitled *Some recent developments and questions on random maps and their scaling limits*. In this series of lectures, he presented a panoramic view of recent results on random maps and their scaling limits. Its first lecture described discrete planar maps and the Brownian map. This lecture was intended as a reminder for students of the basic notions in this field. The second lecture described the recent proof of convergence of the random planar quadrangulations to the Brownian map. The last lecture offered a proof of universality (in terms of the type of random planar maps chosen), as well as a discussion of random maps with higher genius.

Miller and Sheffield jointly presented a four hours mini-course entitled *Imaginary geometry* describing their series of four papers about flow and counterflow lines of the Gaussian Free Field. They showed how many

fundamental problems about regular and whole-plane $SLE(\kappa;\rho_1,\rho_2)$ have natural solutions within this framework. These include questions about time-reversal symmetry, continuity, general boundary duality, dimensions and exponents, CLEs, and other loop ensembles.

2.2 Talks

The random planar maps The first day was mostly devoted to talks on planar maps.

It started with a talk by Le Gall on its joint work with Curien entitled *The harmonic measure of critical Galton-Watson trees*. They consider simple random walk on a critical Galton-Watson tree conditioned to have height greater than n and prove the existence of a constant $\beta \in (0,1)$ such that the hitting distribution of the generation n by random walk is concentrated with high probability on a set of cardinality approximately n^β.

The talk of Curien described the so-called *Random stable looptrees*. These random compact metric spaces are made of a collection of random loops glued together along a tree structure, and can be viewed, in a certain sense, as dual graphs of stable Lévy trees. They are conjectured to arise as universal scaling limits of cluster boundaries in random planar maps decorated with an $O(n)$ model. Curien and Kortchemski proved this conjecture for site-percolation on random triangulations.

Addario-Berry described the *scaling limit of simple triangulations*. This talk was very much related to the mini-course of Miermont since it describes the corresponding result of convergence to the Brownian Map in the case of triangulations with no loops nor multiple edges.

Duplantier talked about *The Hausdorff Dimension of 2D Quantum Gravity*. He studied the Hausdorff dimension D_H of a quantum-gravity random surface with a critical statistical model of conformal central charge $c \in [-2,1]$ (or equivalently $\kappa \in [8/3,8)$) on it. The Knizhnik-Polyakov-Zamolodchikov relation allows one to relate D_H to the Hausdorff dimension of a random map with large faces, recently studied rigorously by Le Gall and Miermont, and reformulated by Borot, Bouttier and Guitter as the gasket of an $O(n)$ loop model on a random quadrangulation.

In his talk, Garban constructed the *Liouville Brownian motion*, a Feller process on the plane or on the sphere which, a.s. in the realization of the Gaussian Free Field X, preserves the Liouville measure $M(dx) = e^{\gamma X}dx$ (with $\gamma < \gamma_c = 2$). He also discussed the possible relevance of this diffusion to the problem of constructing the Liouville metric.

Borot described a more combinatorial approach to random planar maps, through his work *All loop models on the random lattice satisfy a topological recursion*. He enumerated random maps decorated with a loop $O(n)$ model and explained that the enumeration such maps of genus g with n

boundaries, can be settled by a universal recursion on $2g-2+n$. This talk was deeply related to Curien's and Duplantier's talks.

Conformal invariance of the Ising model Several talks were devoted to recent advances in the study of the planar Ising model.

Chelkak presented the derivation, based on the ongoing project with Clément Hongler, of *Exact formulae for spin correlations in the Ising model: holomorphic observables and orthogonal polynomials*. They provide a direct connection between massive holomorphic observables in the planar Ising model and orthogonal polynomials on the unit circle. As a result, they re-derive a number of famous exact formulae for the spin-spin expectations in the full plane (these formulae go back to the fifties).

Cimasoni used the Kac-Ward approach to the Ising model in order to compute the *critical temperature for the Ising model on doubly periodic graphs*.

De Tillière presented a joint project with Boutillier on *Loops in the XOR Ising model* and their relation to GFF. The XOR-Ising model is constructed from two independent Ising models.

Izyurov described *spinor observables*, and their relation to spin correlations and interfaces in the critical Ising model. Among other applications, he proved convergence of interfaces to SLE(3) variants in a general (multiple interfaces in multiply-connected domains) setup.

Kytola presented a new approach to discrete Conformal Field Theory in his work *Discrete holomorphicity, Ising transfer matrix and fermions*. He showed that the square lattice Ising model transfer matrix can be reconstructed from discrete analytic continuation of s-holomorphic (or massive s-holomorphic) functions, and that fermion operators in the transfer matrix formalism are operator valued (massive) s-holomorphic functions.

Conformal Loop Ensembles Several talks described delicate properties of Conformal Loop Ensembles (CLE). The conformal loop ensemble CLE_κ with parameter $\kappa \in (8/3, 8)$ is the canonical family of conformally invariant measures on countably infinite collections of non-crossing loops in a simply connected domain D.

Watson described the *Extremes and the Nesting Field of CLE*. He studied the random set of points in D surrounded by an unusually large or unusually small number of loops asymptotically. He computed the almost-sure Hausdorff dimension of relevant subsets of these points. Finally, using a coupling between the Gaussian free field and CLE_4, he gave a CLE-based treatment of the extremes of the GFF.

Wilson presented a computation of *the Hausdorff dimension of the CLE gasket* (joint work with Jason Miller and Nike Sun). The gasket is the set of points not surrounded by any loop of the CLE. Together with the work of

Schramm-Sheffield-Wilson (2009) giving the upper bound for all κ and the work of Nacu-Werner (2011) giving the matching lower bound for $\kappa \leq 4$, this completes the determination of the CLE_κ gasket dimension for all values of κ for which it is defined.

Wu described a *Conformally Invariant Growing Mechanism in* CLE_4 and *Couplings between GFF and* CLE_4.

Other talks Several talks did not fit in the three main subjects above. Gruzberg presented a new approach to the quantum Hall transition via combinatorics and conformal restriction. He considered a disordered quantum two-dimensional lattice model, the Chalker-Coddington (CC) model, describing the transitions between plateaux in the integer quantum Hall effect. Assuming conformal invariance in the continuum limit, he argued that the conformal restriction theory can be used to describe the so-called disorder-averaged point contact conductances in the model.

Hammond discussed several aspects of the *Self-avoiding walk's endpoint displacement*. In works with Duminil-Copin, and more recently also with Glazman and Manolescu, he obtained rigorous results excluding the extremes of fast and slow behaviour for endpoint displacement.

Manolescu described his recent work with Grimmett on *Percolation on Isoradial Graphs*. The star-triangle transformation is used to obtain an equivalence extending over bond percolation models on isoradial graphs. Amongst the consequences are box-crossing (RSW) inequalities and the universality of alternating arms exponents (assuming they exist) for such models, under some conditions.

Ponsaing talked about *the 2-boundary Brauer model*. The Brauer loop model is an integrable lattice model similar to the fully packed $O(n)$ loop model but with crossing loops allowed.

3 Impact on the field

This workshop was built as a continuation (at a slightly smaller scale), of the successful meeting of Ascona 2010. He gave the opportunity to discuss the recent progress in the field, and to gather scientists from different part of Europe and the world, which do not necessary have the occasion of meeting otherwise. Several recent breakthroughs were discussed actively during the afternoon sessions and the evenings (including the recent progress in the theory of random planar maps). We hope and expect that ideas and strategies exchanged during this workshop will lead to further developments.

The workshop also provided a good summary of the fundamental challenges that the field will have to tackle in the next few years. In particular, several open problems were discussed.

Let us mention that, through mini-courses and discussions, a particular emphasis was made on the transmission of knowledge to youngsters (around 10 Phd students were present during this workshop).

In conclusion, the workshop achieved his main goal: gathering world leaders to try to share ideas and partial advancements. It will be very beneficial for the field, and we expect the impact to be comparable to the one of Ascona's meeting three years ago.

4 Final program

Monday, February 11

- 09:00 – 10:00 *Le Gall* The harmonic measure of critical Galton-Watson trees
- 10:30 – 11:30 *Miermont* Some recent developments and questions on random maps and their scaling limits (1/3)
- 11:30 – 12:30 *Curien* Random stable looptrees
- 14:00 – 16:30 Discussions
- 17:00 – 18:00 *Duplantier* On the Hausdorff Dimension of 2D Quantum Gravity
- 18:00 – 19:00 *Addario-Berry* The scaling limit of simple triangulations

Tuesday, February 12

- 09:00 – 10:00 *Miller / Sheffield* Imaginary Geometry 1: Interacting SLEs
- 10:30 – 11:30 *Miermont* Some recent developments and questions on random maps and their scaling limits (2/3)
- 11:30 – 12:30 *Chelkak* Exact formulae for spin correlations in the Ising model: holomorphic observables and orthogonal polynomials.
- 14:00 – 16:30 Discussions
- 17:00 – 18:00 *Izyurov* Spinor observables, spin correlations and interfaces in the critical Ising model
- 18:00 – 19:00 *Kytola* Discrete holomorphicity, Ising transfer matrix and fermions

Wednesday, February 13

- 09:00 – 10:00 *Miermont* Some recent developments and questions on random maps and their scaling limits (3/3)
- 10:30 – 11:30 *Miller / Sheffield* Imaginary Geometry 2: Reversibility of $SLE(\kappa; \rho_1, \rho_2)$ for $\kappa \in (0, 4)$
- 11:30 – 12:30 *Garban* Liouville Brownian motion
- 14:00 – 16:30 Discussions
- 17:00 – 18:00 *Cimasoni* The critical temperature for the Ising model on doubly periodic graphs
- 18:00 – 18:30 *de Tillière* Loops in the XOR Ising model
- 18:30 – 19:00 *Manolescu* Bond Percolation on Isoradial Graphs

Thursday, February 14

- 09:00 – 10:00 *Wilson* The Hausdorff dimension of the CLE gasket
- 10:30 – 11:30 *Miller / Sheffield* Imaginary Geometry 3: Reversibility of SLE for $\kappa \in (4, 8)$
- 11:30 – 12:30 *Watson* The Extremes and the Nesting Field of CLE
- 14:00 – 16:30 Discussions
- 17:00 – 18:00 *Wu* Conformally Invariant Growing Mechanism in CLE_4 and Couplings between GFF and CLE_4
- 18:00 – 18:30 *Borot* All loop models on the random lattice satisfy a topological recursion
- 18:30 – 19:00 *Ponsaing* The 2-boundary Brauer model

Friday, February 15

- 09:00 – 10:00 *Gruzberg* Combinatorics and conformal restriction in a model of the quantum Hall transition
- 10:30 – 11:30 *Miller/Sheffield* Imaginary Geometry 4: Interior rays, whole-plane reversibility, and space-filling trees
- 11:30 – 12:30 *Hammond* Self-avoiding walk's endpoint displacement