Scientific Report
Workshop on 2D statistical physics

Date: 16 - 21 February 2014
Place: Les Diablerets, SWITZERLAND
Organizer: Stanislav Smirnov

1 Summary

The present workshop continues a tradition of conferences in Les Diablerets organised by the University of Geneva. The aim of these conferences is to bring together researchers in the dynamic field of statistical mechanics to explore the latest advances and future directions of research in the field. Past sessions have been very successful, inciting collaborations and new ideas. The meeting is meant to bring together both young and confirmed researchers in a relaxed environment.

This year the main focus of the workshop was on random maps, the Gaussien Free Field (GFF) and the relation between the two. Two dimensional Liouville quantum gravity (LQG) could be understood as a random metric on the sphere $\mathbf{S}^2$, supposedly obtained by distorting the regular Riemannian metric by an exponential of the GFF (multiplied by a factor $\gamma \in [0,2)$). Moreover the quantum and Riemannian settings should be linked by what is called the Knizhnik-Polyakov-Zamolodchikov (KPZ) relation.

The metric mentioned above is not well-defined mathematically due to the irregularity of the GFF. Various approaches have been proposed to construct and study such an object. Sheffield and others have focused on the study of the GFF, proposing various ways of exploring the GFF that should shed light on Liouville quantum gravity and the KPZ relation. This topic is very dynamic and prolific; mini courses such as the one of Miller and Sheffield allow researchers not directly involved in this research to stay up-to-date with the advances or to penetrate the domain. In addition to the mini course, the talks of Werner, Kemppainen and Aru were relevant for this topic.

Another angle of approach is that of random maps. A random map, viewed as a metric space, could be considered a rough approximation of a random metric on the sphere. Thus, as the map becomes finer, it is expected to converge in the Gromov–Hausdorff sense to the sphere endowed with the afore-mentioned random metric. Recent results of Le Gall and Miermont have indeed proved that certain random maps converge to what is called the Brownian Map, an object expected to be universal and to correspond to LQG with $\gamma = \sqrt{8/3}$. The talks by Le Gall and Miermont were dedicated to the Brownian map and those of Curien and Rhode were relevant to the link between the Brownian map and LQG.

Other subjects were treated in the workshop, such as the scaling limit of critical percolation (Watson and Beffara) and near-critical one (Garban), a characterisation of the possible behaviours of critical FK-percolation (Tassion) and the abelian sandpile model (Kassel). The first talks came as an incentive to revive the attempts to prove conformal invariance for percolation in more general settings. Tassion's presentation is especially worth mentioning, since it gives powerful results for FK-percolation through "classical" techniques that open up many new possibilities for exploring this model. A natural question to ask next is to prove that, for $q > 4$, planar FK-percolation exhibits a first order phase transition.

In conclusion, the meeting was dominated by the subject of planar maps and Liouville quantum gravity. The KPZ relation, which seemed hopelessly out of reach a few years ago, is now much better understood due to the rapid advances in the understanding of the GFF on the one hand and to the fast expansion of the field of random maps on the other. It has been made evident that this topic deserves much attention in the following years.

2 Scientific content

The workshop contained 23 talks ranging between 40 minutes and one hour. Five sessions were dedicated to a mini course by Jason Miller and Scott Sheffield on Quantum Loewner Evolutions.

2.1 Quantum Loewner Evolutions mini course.

The five-session course presented recent results that allow the introduction of a new process called quantum Loewner evolution (or QLE), which may be very useful in studying the GFF.

In some sense the work of Miller and Sheffield originates with the study of the scaling limit of diffusion limited aggregation (DLA) in the plane. This is an old and famously difficult question. One can generalise the question in two ways: first, one may consider the *dielectric breakdown model* η-DBM, a generalisation of DLA in which particle locations are sampled from the η-th power of harmonic measure, instead of harmonic measure itself. Second, instead of restricting attention to deterministic lattices, one may consider η-DBM on random graphs known or believed to converge in law to a Liouville quantum gravity (LQG) surface with parameter $\gamma \in [0, 2]$. In this context, they propose a scaling limit candidate, the quantum Loewner evolution, $QLE(\gamma^2, \eta)$.

Quantum Loewner evolutions are defined in terms of the radial Loewner equation, like radial SLE, except that they are driven by a measure-valued diffusion ν_t derived from LQG, rather than a multiple of a standard Brownian motion. They formalise the dynamics of ν_t using an SPDE Furthermore they find that, for each $\gamma \in (0, 2]$, there are two or three special values of η for which they establish the existence of a solution to these dynamics and explicitly describe the stationary law of ν_t.

They also explained how discrete versions of this construction relate DLA to loop-erased random walk and the Eden model to percolation. A certain "reshuffling" trick (in which concentric annular regions are rotated randomly, like slot machine reels) facilitates explicit calculation.

They proposed $QLE(2, 1)$ as a scaling limit for DLA on a random spanning-tree-decorated planar map, and $QLE(8/3, 0)$ as a scaling limit for the Eden model on a random triangulation. They also suggested using $QLE(8/3, 0)$ to endow pure LQG with a distance function, by interpreting the region explored by a branching variant of $QLE(8/3, 0)$, up to a fixed time, as a metric ball in a random metric space.

2.2 Talks on the GFF and random maps

Wendelin Werner presented an ongoing project with Jason Miller and Scott Sheffield (related to the work they presented at last year's workshop) on the relation between FK-percolation and the Potts model, in the continuous setting of CLE and GFF. His talk touched the relation between the Conformal Loop Ensembles of parameters κ and $16/\kappa$, the conjectural scaling limit of the FK-Potts coupling and of the 'divide-and-color' model and some notion of critical percolation within a CLE carpet.

In his talk, Kemppainen introduced the Conformal Loop Ensemble (CLE) on the Riemann sphere. Previously defined in the plane, CLE may be extended to the sphere, provided it satisfies certain invariance properties. The talk was dedicated to showing that a natural family of nested CLEs in the full plane is invariant under inversion in the plane. His work is joint with Wendelin Werner, and was somewhat related to his talk.

A subject treated in several talks was the one of uniformization of planar maps. Planar maps can be interpreted as Riemann surfaces obtained from gluing polygons together. This provides an essentially unique way (up to Mobius transformations) of drawing the map in the plane. Steffen Rohde studied this construction for trees viewed as a map with a single face, and treated questions regarding the existence of limits of such trees.

Nicolas Curien gave two talks on the conformal structure of random planar maps; deeply related to the mini course by Miller and Sheffield. A uniform random triangulation of the sphere can naturally be considered as a random Riemann surface and one can study its "conformal structure" (as that presented in Rohde's work), which is conjectured to be strongly linked to the Gaussian free field. Curien presented a path to study the conformal structure of random planar maps based on their Markovian exploration by an independent SLE_6 process. Although his results are based on a technical assumption (called the *-hypothesis), which is not yet proved, they give hope that some weak instance of the KPZ relation may be proved. His work is only the beginning of such a proof, but exhibits some very promising ideas.

In the second talk he also discussed a new class of random triangulations of the plane which have a hyperbolic flavour.

In another talk related the GFF and the KPZ relation, Juhan Aru explained how the regular KPZ relation does not hold for sets that, instead of begin independent of the field, are coupled with the GFF. He illustrated this by considering the SLE flow lines of the GFF, for which he proved a formula relating their quantum and regular dimension, which differed from the usual KPZ relation.

The opening talk was given by Jean-Francois Le Gall, who talked about some new distributions for the Brownian map and the UIPT. He presented some explicit calculations of distributions related to the Brownian map and to the uniform infinite planar triangulation and quadrangulation. In particular, in the infinite volume version of the Brownian map, he obtained the distribution of the volume of the extended ball of radius r, and of the associated random process when r varies. He did this by interpreting these distributions in terms of the peeling process associated with the UIPT.

Another talk on the topic of planar maps was that of Gregory Miermont who used a new bijection between maps and trees (the Ambjorn-Budd bijection) to prove that a maps with n edges chosen uniformly at random among all such maps converges when properly rescaled to the Brownian map. This extends previous results by Le Gall and Miermont.

2.3 Other talks

Sam Watson has discussed a recent result on the rate of convergence in Cardy's formula. A celebrated theorem by Smirnov shows that crossing probabilities of fixed domains for site percolation on the triangular lattice converge, as the mesh-size of the lattice goes to 0, to some quantity, obtained by the so-called Cardy formula. Watson, jointly with Dana Mendelson and Asaf Nachmias, have obtained explicit values for the rate-of-convergence exponents for piecewise analytic Jordan domains.

In a presentation related to Watson's one, Vincent Beffara exposed his recent work aimed at extending the Cardy-Smirnov theorem to other percolation models than just site percolation on the triangular lattice. The only known proof of the theorem is very specific to the triangular lattice, hence not allowing to test the universality assumption that states that the large scale behaviour of critical percolation does not depend on the microscopic details of the lattice. The models considered in this talk are an interpolation between the triangular lattice and general planar graphs.

Tassion discussed some recent exciting results on the critical behaviour of planar FK-percolation. His work with H. Duminil-Copin and V. Sidoravicius, proves that, depending on q, there are two possible behaviours for critical FK-percolation on the square lattice. One corresponds to the existence of a unique infinite-volume measure which exhibits a strong form of RSW (crossing estimates); in the second there are multiple infinite-volume measures, the extremal ones exhibiting some form of exponential decay. The present work, in conjunction with previous results by Hugo Duminil-Copin, shows that when $1 \leq q \leq 4$, the first scenario occurs. The second is expected to occur for $q > 4$, but no proof is available yet. This confirms physics predictions for the Potts model.

Christophe Garban talked about the scaling limit of the Minimal Spanning Tree in the plane. He has talked about several models of Minimal Spanning Trees, the original being the following. Sample N points uniformly in a unit square of the plane. Among all the spanning trees which cover these N points, consider the tree with minimal euclidean length, called the Minimal Spanning Tree MST_N. The aim of his talk was studying the geometrical propertied of this object as N tends to infinity.

He explained how to construct a "continuous tree" which should be the scaling limit, as N goes to infinity, of the MST_N. Nevertheless, in the above "Poissonian" definition a planar Minimal Spanning Tree, this limiting behaviour still remains conjectural. But for an analogous model on a well chosen planar graph, much more can be done. In a long-term project with Gabor Pete and Oded Schramm, only recently completed, they have proved that the Minimal Spanning Tree defined on the triangular lattice (by considering the lengths of the edges of the triangular lattice are i.i.d. exponentials) converges to a continuous tree of a new kind, seemingly different from the objects expected as scaling limits of other models. Standard universality arguments suggest that the limiting tree should not depend on the microscopic structure of the model.

Adrien Kassel introduced a model less familiar to the audience, namely the abelian sandpile model. For a given (finite) graph G the model may be viewed as a Markov chain, in which grains of sand are added at exponential times to the sites of the lattice and as soon as a site contains more grains than it has neighbours, it instantly distributes one to each of its neighbours. Finally the system contains a sink that absorbs the excess, thus stabilising the system until a new particle is added. The recurrent configurations of this Markov chain are in bijection with the spanning trees of G.

The sandpile model may also be defined on many infinite lattices, and is in close connection with the uniform spanning forest of such lattices. A quantity of interest is the "sandpile density", which is the average amount of sand per vertex under the invariant measure of the Markov chain discussed above.

Kassel discussed his recent work with David Wilson that allows to compute sandpile densities for several planar lattices using a notion of planar duality. On the square lattice this quantity had been conjectured to be equal to 17/8 (Grassberger in the 90's), a result which was only proved in 2011 by methods which do not extend well to other lattices.

Other talks included Bertrand Duplantier's presentation of a phase transition for the multifractal spectrum of SLE curves; Manolescu's talk on self-destructive percolation and forest fires on planar lattices; a presentation by Izyurov of improvements of older results on the conformal invariance of spin-energy correlations in the critical Ising model and a talk by Igor Kortchemski about limits of random dissections of polygons with Boltzmann weights.

3 Results and impact

In continuing a series of similar workshops in Les Diablerets (also worth mentioning is the one in Ascona in 2010), the present meeting contributes to creating an important international centre for statistical mechanics in Geneva, closely linked to the universities and schools of Lyon and Zurich. Most of the young participants came from these universities, the more senior researchers were from across Europe, some even from the US.

The presentations summarised recent advances in the field of two dimensional statistical mechanics and emphasised some new, very promising directions of research. This allowed established researchers as well as young ones (around 15 PhD students were present) to observe the latest results and recap the main open questions of the field.

Time was dedicated to informal discussions in a hope to establish future collaborations. This type of gathering of experts in a relatively small and relaxed workshop already proved to be fruitful (as past meetings show), and we are confident it will continue to do so.

Annex: programme

Sunday, February 16

- 09:00 - 10:00 *Le Gall:* Some new distributions for the Brownian map and the UIPT
- 10:30 - 11:30 *Sheffield/Miller:* Quantum Loewner Evolution (mini course)
- 11:30 - 12:30 *Garban:* Scaling limit of the Minimal Spanning Tree in the plane
- 17:15 - 18:00 *Curien:* On the conformal structure of random planar maps I
- 18:00 - 18:45 *Watson:* Rate of convergence in Cardy's formula

Monday, February 17

- 09:00 - 10:00 *Tassion:* On the critical behavior in planar FK-percolation
- 10:30 - 11:30 *Sheffield/Miller:* Quantum Loewner Evolution (mini course)
- 11:30 - 12:30 *Werner:* About the FK-Potts relation in the continuous CLE/GFF setting
- 17:15 - 18:00 *Sheffield/Miller:* Quantum Loewner Evolution (mini course)
- 18:00 - 18:45 *Curien:* On the conformal structure of random planar maps II

Tuesday, February 18

- 17:00 - 17:40 *Kemppainen:* CLE on Riemann sphere
- 17:40 - 18:20 *Kassel:* The sandpile density of planar graphs
- 18:20 - 19:00 *Aru:* SLE + GFF != KPZ

Wednesday, February 19

- 09:00 - 10:00 *Beffara:* Percolation on mesoscopic lattice
- 10:30 - 11:30 *Sheffield/Miller:* Quantum Loewner Evolution (mini course)
- 11:30 - 12:30 *Miermont:* The scaling limit of uniform random maps, via the Ambjorn-Budd bijection
- 17:15 - 18:00 *Kortchemski:* The Brownian Continuum Random Tree is the scaling limit of random dissections
- 18:00 - 18:45 Izyurov Mixed correlations in the critical Ising model

Thursday, February 20

- 09:00 - 10:00 *Duplantier:* Multifractality of Whole-Plane SLE
- 10:30 - 11:30 *Sheffield/Miller:* Quantum Loewner Evolution (mini course)
- 11:30 - 12:30 *Rohde:* How to draw trees
- 17:15 - 18:00 *Manolescu:* Planar lattices do not recover from forest fires
- 18:00 - 18:45 *Kleptsyn:* Quantum gravity and (min,+)-type recursive distributional equations

Annex: list of participants

Juhan Aru (ENS, Lyon) - speaker
Vincent Beffara (Max Planck, Bonn) - speaker
Ilia Binder (University of Toronto)
Roman Boykiy (Université de Genève)
Andrey Bytsko (Université de Genève)
Dmitry Chelkak (Chebyshev Laboratory, St.Petersburg)
David Cimasoni (Université de Genève)
Loren Coquille (University of Bonn)
Nicolas Curien (CNRS - Paris 6) - speaker
Bertrand Duplantier (CEA, Saclay) - speaker
Maxime Gagnebin (Université de Genève)
Christophe Garban (ENS, Lyon) - speaker
Sasha Glazman (Université de Genève)
Konstantin Izyurov (University of Helsinki) - speaker
Adrien Kassel (ETH, Zurich) - speaker
Antti Kemppainen (University of Helsinki) - speaker
Misha Khristoforov (Université de Genève)
Victor Kleptsyn (CNRS, Rennes) - speaker
Igor Korchemski (ENS, Paris) - speaker
Jean-Francois Le Gall (Université Paris-Sud) - speaker
Jhih-Hunag Li (Université de Genève)
Titus Lupu (Université Paris-Sud)
Ioan Manolescu (Université de Genève) - speaker
Gregory Miermont (ENS, Lyon) - speaker
Jason Miller (MIT) - speaker
Anhminh Pham (Université de Genève)
Wei Qian (ETH, Zurich)
Steffen Rohde (University of Washington) - speaker
Ron Rosenthal (ETH, Zurich)
Marianna Russkikh (Chebyshev Laboratory, St.Petersburg)
Scott Sheffield (MIT)
Stanislav Smirnov (Université de Genève)
Daria Smirnova (Université de Genève)
Tatiana Smirnova- Nagnibeda (Université de Genève)
Vincent Tassion (ENS, Lyon) - speaker
Michele Triestino (ENS, Lyon)
Sam Watson (MIT) - speaker
Wendelin Werner (ETH, Zurich) - speaker