



Research Networking Programmes

Short Visit Grant or Exchange Visit Grant

(please tick the relevant box)

Scientific Report

Scientific report (one single document in WORD or PDF file) should be submitted online within one month of the event. It should not exceed eight A4 pages.

Proposal Title: Some conjectures on the topological complexity and related invariants

Application Reference N°: 5831

1) Purpose of the visit

In January 2013, L. Vandembroucq, J. Carrasquel and J. Calcines started in Braga some discussions about both Iwase-Sakai's conjecture and Doeraene-El Haouari's conjecture. Such discussions have become a source of research during the present year that have led to several interesting results. The purpose of the visit to Braga was to complete and finish these mathematical discussions as well as to write down, if possible, the results obtained so far.

2) Description of the work carried out during the visit

The notion of topological complexity of a space $TC(X)$ was introduced by M. Farber in 2003 motivated by the problem of finding the smallest number of continuous instructions for a robot to move from one state to another. Farber also introduced a new invariant restricting motions by giving certain additional conditions to the movements and called it the symmetrical topological complexity. Such a stronger invariant suggested N. Iwase and M. Sakai another motion planning under the condition that a motion is stasis if the initial and the terminal states are the same. This other new invariant is called the monoidal topological complexity, denoted as $TC^M(X)$. In a first state of Iwase-Sakai's research they established that $TC(X)=TC^M(X)$ when X is a locally finite simplicial complex. Unfortunately a mistake was detected in their proof and at present this equality has become a conjecture. Recently, A. Dranishnikov presented some results supporting the Iwase-Sakai conjecture in an unpublished work but so far such a conjecture still remains as an open problem. On the other hand Doeraene and El Haouari introduced an

approximation of $\text{secat}(f)$ the sectional category (or Schwarz genus) called relative category (denoted as $\text{relcat}(f)$) and proved that the difference between the sectional category and the relative category is at most one. Then they asked for which cases both invariants agree. As they checked, in general, these two invariants do not agree. However Doeraene-El Haouari conjectured that the equality $\text{secat}(f)=\text{relcat}(f)$ holds as long as the map has a homotopy retraction.

The work carried out in this visit was two-fold. On the one hand, we did daily work sessions in order to obtain lower bounds for sectional category and relative category for which the corresponding versions of the two conjectures (Iwase-Sakai's and Doeraene-El Haouari's) are true. On the other hand we also wrote down the results that we have obtained throughout the year.

3) Description of the main results obtained

The main results obtained are the following:

We have discovered that the monoidal topological complexity of a space X is the relative category of its diagonal map when X is locally equiconnected (i.e. the diagonal map is a closed cofibration). As a consequence we have that Doeraene-El Haouari's conjecture contains Iwase-Sakai's conjecture. Another important consequence is the fact that $\text{TC}(X)=\text{TC}^M(X)=\text{cat}(X)$ when X is a connected H -space (here $\text{cat}(X)$ denotes the Lusternik-Schnirelmann category of X). This last consequence is a slight improvement of a result given by Dranishnikov.

By considering the k -th iteration of the suspension functor we obtain stable approximations of both the sectional category and the relative category so we can establish a stable version of Doeraene-El Haouari conjecture. We have proved that, when f admits a homotopy retraction, then such stable approximations are the same. In other words, the stable Doeraene-El Haouari conjecture is, actually, a theorem. As a consequence we obtain a positive answer to the stable version of Iwase-Sakai's conjecture. That is, the stable versions of topological complexity and the monoidal topological complexity are the same invariants.

We introduced a weak version of relative category in the sense of Berstein-Hilton. In order to do this we have established a more manageable characterization of relative category when the map under consideration is a closed cofibration. Comparing this weak relative category with the existing weak sectional category one can introduce a weak version of Doeraene-El Haouari's conjecture. We have established a positive answer to such weak version. That is, when the map has a homotopy retraction, then the weak relative category and the weak sectional category of the map agree. As a consequence the weak topological complexity and the weak monoidal topological complexity agree when the space under consideration is locally equiconnected.

We have also faced Doeraene-El Haouari's conjecture in rational homotopy theory. If $f:Y \rightarrow X$ is a map between 1-connected spaces of finite type over the rational numbers and we consider the rationalization f_0 , then the rational Doeraene-El Haouari's conjecture reads as: if f admits a homotopy retraction, then $\text{secat}(f_0)=\text{relcat}(f_0)$. Using Carrasquel's characterization of the sectional category $\text{secat}(f_0)$ in terms of any surjective model of f (in the category cdga of commutative differential graded algebras)

we have obtained a partial result about the equality $\text{relcat}(f_0)$ and $\text{secat}(f_0)$ involving the nilpotency of the kernel of any surjective cdga model for f . Namely, if $\phi:A\rightarrow B$ is a surjective cdga model for f with $K=\text{Ker}(\phi)$, then $\text{relcat}(f_0)$ is less than, or equal to $\text{nil}(K)$. In particular, if (A,d) is a cdga model for X , then $\text{TC}_0^M(X)$ is less than, or equal to $\text{nil}(\ker \mu_A)$, where μ_A is a cdga model of the diagonal map $\Delta:X\rightarrow X\times X$. Using this result together with some other results due to B. Jessup, A. Murillo and P.-E. Parent we can exhibit two important classes of spaces for which the rational version of Iwase-Sakai's conjecture is true. Namely, the 1-connected formal spaces and the 1-connected spaces such that its rational homotopy is of finite dimension and concentrated in odd degrees.

4) Future collaboration with host institution (if applicable)

I expect to continue the collaboration with professor L. Vandembroucq (Braga) about these topics on (monoidal) topological complexity, and sectional category and its relative counterpart in a coming future. In this sense, during this visit to Braga we have established several other discussions concerning, for instance, an open-cover characterization of relative category that might lead to interesting consequences in monoidal topological complexity.

5) Projected publications / articles resulting or to result from the grant (*ESF must be acknowledged in publications resulting from the grantee's work in relation with the grant*)

It is expected to write down our results in a paper that would be submitted to a specialized mathematical journal, possibly the Proceedings of the American Mathematical Society, or Topology and its Applications. The tentative title of the paper is "Relative category and monoidal topological complexity", by J.G. Carrasquel-Vera, J.M. Garcia Calcines and L. Vandembroucq.

6) Other comments (if any)