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Scientific Report

Scientific report (one single document in WORD or PDF file) should be submitted online within one month of the event. It should not exceed eight A4 pages.

Proposal Title: Lusternik-Schnirelmann category of Lie groups and symmetric spaces

Application Reference N°: 6442

SCIENTIFIC REPORT

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Purpose of the visit A stay of one week (from 9 to 15 March 2014) at the University of Lille 1, France.

I visited Villeneuve d'Ascq (Cédex) together with Professor E. Macías-Virgós (University of Santiago de Compostela, Spain) to work with Professor D. Tanré (University of Lille 1). The purpose of my visit was to make progress on the computation of the Lusternik-Schnirelmann category of Lie groups and symmetric spaces. Our goal was to use Morse-Bott theory to obtain an upper bound for this invariant in the case of quaternionic Stiefel manifolds.

My travel and subsistence expenses were covered by an ESF ACAT Short Visit Grant (ref. 6442).

Description of the work carried out during the visit One of the classical problems proposed by T. Ganea in 1971 was to compute the Lusternik-Schnirelmann category for Lie groups and homogeneous spaces [3]. However only a small number of calculations have been carried out. This is particularly true for the symplectic group where the only known results are $\operatorname{cat} Sp(2) = 3$, obtained by P. Schweitzer in 1965 [12], and $\operatorname{cat} Sp(3) = 5$, proved by L. Fernández, A. G. Tato, J. Strom and D. Tanré in 2001 [2] and by N. Iwase and M. Mimura [5]. The latter authors also proved that $\operatorname{cat} Sp(n) \geq n + 2$ for $n \geq 3$. The conjecture made by M. Hunziker and M.R. Sepanski in 2009 [4] should imply that $\operatorname{cat} Sp(n) \leq \lfloor \frac{(n+2)^2}{4} \rfloor - 1$.

The progress in the case of Stiefel manifolds is also so slow. In 1976, W. Singhof [13] proved that the category of real Stiefel manifolds is $\operatorname{cat}(W_{n,k}) = k$. His technique consists mainly in considering certain contractible subsets associated to matrix eigenvalues. In 2011, H. Kadzisa and M. Mimura used Morse-Bott functions to obtain the category of some real and quaternionic Stiefel manifolds of specific orders [6]. T. Nishimoto employed in 2007 Singhof's idea to get that $\operatorname{cat} X_{n,k} = k$ for $n \geq 2k$ [11].

In [8] we showed that $\operatorname{cat} Sp(n) \leq \binom{n+1}{2}$ by using some Morse-Bott functions. This was achieved by computing the number of critical levels of such functions. We expect the same techniques could be extended to Stiefel manifolds. So, first of all, coming back to the original idea of Lusternik and

Schnirelmann, we were trying to bound the category of Stiefel manifolds $X_{n,k} = Sp(n)/Sp(n-k)$ by computing the number of critical points. We chose height functions $h_\omega \colon \mathbb{H}^{n\times k} \to \mathbb{R}$ with respect to the hyperplane orthogonal to a given matrix $\omega * = (\delta \ D_k)$ with blocks of size $k \times (n-k)$ and $k \times k$ respectively. Then, we consider its restriction to $X_{n,k}$, f_ω, where $f_\omega \begin{pmatrix} T \\ P \end{pmatrix} = \Re\mathrm{Tr}(\delta T + DP)$ for any $\begin{pmatrix} T \\ P \end{pmatrix} \in X_{n,k}$. The solutions of the critical equation are points with $P = D^*(\delta\delta^* + DD^*)^{-1}Y$ where Y is any matrix such that $Y^2 = \Delta$. When Δ has different positive eigenvalues, there are only a finite number of matrices Y. However, even in this case, the number of critical levels is too big. The bound we can give in this simple way is $\operatorname{cat} X_{n,k} \leq \binom{k+1}{2}$.

Our second effort was to extend to the general case the ideas developed by Nishimoto for $n \geq 2k$. He considers as open contractible subset the interior of a maximal dimension cell and moves it by homeomorphisms to cover the whole space. We think this cell can be proven to be contractible in an analogous way as we did for Lie Groups and symmetric spaces [9]. Our technique is based on the so-called Cayley transform with values in a vector space. Unfortunately, we proved that it's not possible to extend Cayley map in a natural way. As we need the image to be in the tangent and with some block that has to be skew-hermitian, then the map has to be null. However, our ideas could be related to that on papers by H. Miller [10] and N. Kitchloo [7], which deserve further study.

Professor E. Macías-Virgós, Professor D. Tanré and me worked mainly at the *Laboratoire Paul Painlevé* (University of Lille 1). The Laboratoire made available to us its library and rooms with blackboard to discuss. We worked around eight hours per day, form 9:00 to 19:00, with some breaks for lunch or coffee.

Description of the main results obtained The bound we can give by counting the critical points of a height function is $\operatorname{cat} X_{n,k} \leq \binom{k+1}{2}$. The advance is that we don't need to distinguish between $n \geq 2k$ and $n < 2k$ situations. A similar result had been announced by E. Macías-Virgós, J. Oprea, J. Strom and D. Tanré, but we gave a complete correct proof.

On the other hand, the relationship between the results of Nishimoto and Cayley transform can be applied to obtain stable splittings of Stiefel manifolds.

Future collaboration with host institution (if applicable) Our collaboration with Prof. Tanré of *Laboratoire Paul Painlevé* will continue thoroughout next months. He will visit Santiago de Compostela next summer and we plan to come back to Lille in a future occasion. We think we can give a more tight upper bound for $\operatorname{cat} X_{n,k}$ and this result will allow us to compute $\operatorname{cat} Sp(n)$.

Also, we have some ideas to get the last one by computing the relative category of the Grassmannians that appear in the critical levels of Morse-Bott functions.

At last, we hope we can work on Cayley transform to find some way to construct it on Stiefel manifolds.

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