An acyclic subspace approach to the computation of the homomorphism induced in homology by a correspondence

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May 30, 2015

Purpose of the visit. The purpose of this short visit was to investigate, in collaboration with Marian Mrozek, the possibility of improving the effectiveness of algorithmic computation of the homomorphism induced in homology by a correspondence by applying the approach based on the construction of an acyclic subspace.

Work carried out. We discussed the basis for the method, which consists of the three publications [2], [1], and [3]. We discussed the conditions that must be satisfied by the constructed acyclic subsets, an algorithm for achievieng the goal, as well as possible advantages and bottlenecks of different strategies of conducting the construction.

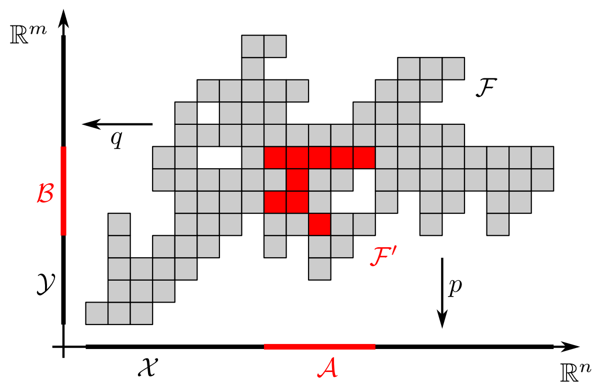

Figure 1: Construction of an acyclic subspace.

Results. We developed the following general method for the construction of acyclic subspaces in the graph of the correspondence, as well as in its domain and codomain; in the description, we follow the notation introduced in the publications mentioned above; see also Figure 1. Given a combinatorial cubical representation of a correspondence (e.g., a combinatorial cubical multivalued map) $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{Y}$, one can construct an acyclic subset $\mathcal{F}'$ of $\mathcal{F}$ together with the corresponding acyclic subsets $\mathcal{A} \subset \mathcal{X}$ and $\mathcal{B} \subset \mathcal{Y}$ using the following procedure. One first selects an arbitrary element $Q_0 \in \mathcal{F}$, and this choice determines its projections $p(Q_0) \in \mathcal{X}$ and $q(Q_0) \in \mathcal{Y}$. These three grid elements form the initial acyclic subsets:

$$\begin{aligned}
\mathcal{F}' &:= \{Q_0\} \subset \mathcal{F}, \\
\mathcal{A} &:= \{p(Q_0)\} \subset \mathcal{X}, \\
\mathcal{B} &:= \{q(Q_0)\} \subset \mathcal{Y}.
\end{aligned}$$

Then the sets $\mathcal{F}' \subset \mathcal{F}$, $\mathcal{A} \subset \mathcal{X}$ and $\mathcal{B} \subset \mathcal{Y}$ are grown as much as possible, in the following way. Each cubical grid element $Q \in \mathcal{F} \setminus \mathcal{F}'$ is checked whether the following three inclusions induce isomorphisms in homology:

$$\begin{aligned} |\mathcal{F}'| &\hookrightarrow |\mathcal{F}' \cup \{Q\}|, \\ |\mathcal{A}| &\hookrightarrow |\mathcal{A} \cup \{p(Q)\}|, \\ |\mathcal{B}| &\hookrightarrow |\mathcal{B} \cup \{q(Q)\}|. \end{aligned}$$

This condition is verified by analysing the collection of neighbors of the grid elements that are about to be added, and this verification is relatively efficient. If this condition is satisfied then one adds Q to $\mathcal{F}'$, $p(Q)$ to $\mathcal{A}$, and $q(Q)$ to $\mathcal{B}$. When no more cubes in $\mathcal{F} \setminus \mathcal{F}'$ can be added to $\mathcal{F}'$ then—as the last step—the sets $\mathcal{A}$ and $\mathcal{B}$ can be grown in $\mathcal{X}$ and $\mathcal{Y}$, respectively, without the constraint related to the graph of the map.

After having conducted this construction, one can compute the homomorphism induced in relative homology by the pair of combinatorial cubical correspondences $(\mathcal{F}, \mathcal{F}') \colon (\mathcal{X}, \mathcal{A}) \rightrightarrows (\mathcal{Y}, \mathcal{B})$, as in [1]. Except for the 0th homology level, the computed homomorphism is going to coincide with the one induced by $\mathcal{F} \colon \mathcal{X} \rightrightarrows \mathcal{Y}$. Note that the homomorphism at level 0 can be easily determined by checking how the connected components are mapped to each other.

Future collaboration. Although the choice of the initial cube $Q_0 \in \mathcal{F}$ is arbitrary and in general should not affect the quality of the algorithm, determining an optimal order in which the other cubes $Q \in \mathcal{F} \setminus \mathcal{F}'$ are taken into consideration seems to be a nontrivial task. A suitable strategy must be developed, whose one of the main aims should be maximizing the sizes of the constructed acyclic subsets, so as to provide an optimal improvement in homology computation speed. It is also not obvious how large the sets can be constructed in practical situations; if the structue of the map limits the possibility of adding cubes to the acyclic sets then the gain might turn out to be much smaller than in other cases.

We plan to implement the algorithm and try several strategies for the construction of the acyclic subsets. We are going to test the method on a selection of various types of correspondences, and to measure the gain achieved in those examples.

Projected publications. We expect to describe the method and the results of the tests in a research article, which will be submitted for publication in an international peer-refereed academic journal in the future.

References

[1] S. Harker, H. Kokubu, K. Mischaikow, P. Pilarczyk, Inducing a map on homology from a correspondence, *Proc. Amer. Math. Soc.*, accepted (2015).

[2] K. Mischaikow, M. Mrozek, P. Pilarczyk, Graph approach to the computation of the homology of continuous maps, *Foundations of Computational Mathematics* 5 (2005), 199–229.

[3] N. Żelazna, *Algorytmy homologii zbiorów kostkowych metodą podzbioru acyklicznego* [Algorithms for the homology of cubical sets using the method of an acyclic subset], PhD thesis [in Polish].