Research Networking Programmes

Short Visit Grant ☒ or Exchange Visit Grant ☐

(please tick the relevant box)

Scientific Report

Scientific report (one single document in WORD or PDF file) should be submitted online <u>within one month of the event</u>. It should not exceed eight A4 pages.

<u>Proposal Title</u>: RiP Research stay at Oberwolfach

<u>Application Reference N°</u>: 5890

1) Purpose of the visit

An invitation for a so-called "Research in Pairs no. 1337p" for two weeks (8–21 September 2013) was granted by the Mathematisches Forschungsinstitut Oberwolfach, Germany, to the applicant (Enrique Macías- Virgós, Universidade de Santiago de Compostela, Spain) jointly with John Oprea (Cleveland State University, Ohio, USA), Jeff Strom (Western Michigan University, USA) and Daniel Tanré (Université de Lille 1, France).

While the Institute provided office space, accommodation (full board and lodging), and an allowance for incidental expenses, my travel expenses were covered by an ESF ACAT Short Visit Grant (ref. 5890).

The focus of the "Research in Pairs" (RiP) program by MFO is to bring together small groups of researchers to carry out joint research in the Oberwolfach Institute. Projects from all areas of mathematics can be supported by this programme. The stay of a research "pair", which means at least two and at most four people, can last between two weeks and one month. It is required that not all come from the same place.

The ultimate goal of our work was to determine the LS-category of the symplectic group Sp(n) of quaternionic matrices $A \in H^{(n\times n)}$

such that $AA^* = I$ and that of the Stiefel manifolds $X_{n,k}= Sp(n)/Sp(n-k)$. The four participants had obtained previous results related to this problem.

2) Description of the work carried out during the visit

For a general overview on LS-category see [CLOT03]. A famous series of problems on LS-category was listed in [Gan71]. Many questions from this list are still open and among them is the determination of the LS-category of Lie groups. The main result in this direction has been the determination of $\operatorname{cat} U(n)=n$ and $\operatorname{cat} SU(n)= n-1$ [Sin75]. But, for the symplectic groups Sp(n), and the rotation groups SO(n) there are only partial results. For instance, in the case of Sp(n) we only know that $\operatorname{cat} Sp(2) = 3$ [Sch65] and $\operatorname{cat} Sp(3) = 5$ [FSGTST04]. For SO(n) the situation is analogous [IKM07]. Some preliminary bounds have also been determined: for instance, $\operatorname{cat} Sp(n) \geq n+2$ when $n \geq 3$ [IM04] and $\operatorname{cat} Sp(n) \leq n(n+1)/2$ for all n [MVPS13].

Our goal was to compute cat Sp(n) for some new values of n and to establish better lower and upper bounds of cat Sp(n). We planned to approach the problem in two different ways: from a linear algebra viewpoint [MVPS09, MVPS10] akin to Singhof's method for the unitary group; and from a homotopy-theoretic viewpoint represented by [FSGTST04, Str03]. In practice we also revised many papers from the literature and we tried alternative ideas, for instance Morse-Bott theory.

In average we worked together 8-9 hours per day, distributed as follows: (8-9h Breakfast) / 9:30 to 12:30 / (12:30-13:30 Lunch) / 15:30 to 18:30 / (18:30-19:30 Dinner) / 20:00 to 22:00. The MFO put at our exclusive disposal a room with all facilities needed (blackboard, printers) and its excepcional library.

3) Description of the main results obtained

Here is a short list of our main conclusions.

a) Let $X=X_{n,k} =Sp(n)/Sp(n-k)$ be the quaternionic Stiefel manifold of k-frames in H^n. Then
(1) $\operatorname{cat}(X) = k$ when $n \geq 2k$;
(2) $\operatorname{cat}(X)+1\leq(n-k+1)((k+1)(k+2)+1)$ when $n<2k$.
The idea of the proof is to consider the height function $h: H^{n\times k} \to R$ with respect to the hyperplane orthogonal to a given matrix ω. Then $h(x) = \operatorname{ReTr}(\omega^* x)$, up to the constant $|\omega|$. We choose the matrix ω in such a way that the restriction f of h to X has isolated critical points

and few critical levels. Then we apply the result cat $(M)+1 \leq \#\{$critical values of $f\}$ [RS03]). Although part (1) is already known [Nis07], our method is completely different. The second part is new.

b) Instead of computing cat G we tried to compute the topological complexity TC(G), which equals the former one for any Lie group G [Far03]. We proved that for all quaternions α, β of norm 1 the open set $\Omega(\alpha,\beta) = \{(A, B) \in G \times G: A\alpha + \beta B$ is invertible$\}$ is categorical in the sense that there exists a section of the path fibration $G^I \to G\times G$. This seems to be the correct generalization of Singhof's proof for U(n).

c) The critical sets of a Morse-Bott function on G are Grasmannians of different dimensions. The normal (unstable and stable) bundles allow to reconstruct the Lie group in a precise manner [Mil85]. This is a generalization of the cellular structure determined by a Morse function. The study of the homotopy push-outs associated to this construction seems to give an upper bound for the weak category of G. In particular we conjecture that wcat $Sp(4) \leq 7$. The same result should follow from a detailed study of the so-called "category sequences" [NSS06].

d) It is possible to associate to any quaternionic matrix A a characteristic polynomial L(t) which detects the complex left eigenvalues of A. This allows to reduce the computation of cat Sp(n) to the category of the set Z of matrices such that $L(t) = 0$ for all $t \in C$. The characterization of those matrices belonging to Z is still an open problem.

e) We elaborated several computer algorithms related to the previous work.

4) Future collaboration with host institution (if applicable)

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5) **Projected publications / articles resulting or to result from the grant** ***(ESF must be acknowledged in publications resulting from the grantee's work in relation with the grant)***

The MFO publishes a preprint series Oberwolfach Preprints (OWP), which mainly contains research results related to a longer stay in Oberwolfach. In particular, this concerns the Research in Pairs Programme (RiP).

We intend to publish there (and then to send to a journal, most probably "Proceedings of AMS") a paper entitled "LS category of quaternionic Stiefel manifolds" containing our results. We also plan to submit another paper on some applications and computer algorithms related to LS category to the journal Applicable algebra in engineering, communication and computing which is preparing an special issue on “Computer algebra in algebraic topology and its applications”. The editors of this volume will be G. Ellis, E. Sáenz-de-Cabezón, A. Murillo and P. Real.

In all those publications we shall include an acknowledgement to ESF ACAT by its finantial support.

6) Other comments (if any)

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