CAST TRAVEL GRANT REPORT
ENERGY-CAPACITY INEQUALITIES AND SPECTRAL INVARIANTS FOR COISOTROPIC SUBMANIFOLDS AND DISPLACEABLE LAGRANGIAN SUBMANIFOLDS

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The following is the scientific report for the CAST short visit grant for my visit to the University of Nantes from July 15th to July 30th, 2013 to work with Samuel Lisi.

1. Purpose of the visit

The original purpose of this visit was to continue work on a Hofer-Zehnder type capacity $c(M, N)$ defined relative to co-isotropic submanifolds that we had developed earlier. We had established an energy-capacity inequality of the form $c(M, N \cap M) \leq \alpha e(N, M)$, where $\alpha > 0$, N here is a Lagrangian submanifold such that the Lagrangian spectral invariants of Leclercq [2] are well-defined, i.e. $\mu|_{\pi_2(M,L)} = 0$ and $\omega|_{\pi_2(M,L)} = 0$, and $e(N, M)$ is the minimal energy required to displace N from M. Our hope was to extend a version of this inequality to displaceable monotone Lagrangian submanifolds. That is, we conjecture that, in this case, the capacity $c(M, L)$ satisfies $c(M, L) \leq \alpha E(L)$, where $E(L)$ is the displacement energy of the Lagrangian.

2. Work carried out during the visit, main results obtained

We decided that it would be best during this visit to prepare as many of our earlier results for publication as possible, in addition to pursuing new questions about displaceable Lagrangians. In earlier work, we had constructed a Hofer-Zehnder type capacity $c(M, N)$ for pairs (M, N) where $N \hookrightarrow M$ is a properly embedded coisotropic submanifold. The definition is analogous to the definition of the Hofer-Zehnder capacity [1], and we showed that $c(B(1), B_N(1)) = c(Z(1), Z_N(1)) < \infty$, where $B_N(1)$ and $Z_N(1)$ are the $n+k$-dimensional 'coisotropic parts' of $B(1)$ and $R^{2n-2} \times B^2(1)$, respectively, where the balls are considered to be centered at the origin in $\mathbb{C}^n$. In addition, we have established several cases of an energy-capacity inequality for this capacity.

Our main effort during this visit was spent doing two complementary tasks, 1) preparing our original construction of the relative Hofer-Zehnder capacity for publication, and 2) proving lemmas and propositions necessary to show the following non-squeezing conjecture :

Conjecture 1. *Consider* $(\mathbb{R}^{2n}, \omega_0)$*, with* ω_0 *the standard symplectic structure. Let* $a := (0, \dots, 0, y_n)$*,* $a' := (0, \dots, 0, y'_n)$*, let* $B_a(1)$ *denote the unit ball centered at* a*, and let* $B_a^{n,k}(r) \subset \mathbb{R}^{2n}$ *be the coisotropic ball obtained by intersecting the plane*

$$\mathbb{R}^{n,k} := \{x \in \mathbb{R}^{2n} | x = (x_1, \dots, x_n, y_1, \dots, y_k, 0, \dots, 0)\}$$

with $B_a(1)$*. Denote by* $Z_{a'}(1) = \{z \in \mathbb{R}^{2n} \mid x_n^2 + (y_n - y'_n)^2 < 1\}$ *the unit symplectic cylinder centered at* a'*, and let*

$$Z_{a'}^{n,k}(R) = \mathbb{R}^{n,k} \cap Z_{a'}(1).$$

If there exists a relative symplecitic diffeomorphism $\phi : (B_a(1), B_a^{n,k}(r)) \to (Z_{a'}(1), Z_{a'}^{n,k}(R))$*, then* $R > r$*.*

The above, if true, would give a very nice coisotropic counterpart to the Gromov non-squeezing theorem. The methods we have used to try to prove this are variational, and largely follow the main ideas in the construction of our relative Hofer-Zehnder capacity. The estimate of the upper bound, however, is considerably more subtle. During this visit, we have found most of a proof of this result, and we are currently trying to fill a gap in what we believe is the final lemma necessary to establish this conjecture.

Returning to the original purpose for the visit, we began studying a simple spectral-type invariant for displaceable monotone Lagrangians, constructed from the inverse PSS map $\Phi_{PSS}^{-1} : HF_*(L) \to H_*(L)$ for Hamiltonians H that do not displace the Lagrangian. We are able to show that the quantity we have defined gives the energy-capacity inequality $c(M, L) \leq \alpha E(L)$ for some $\alpha > 0$ if it is also invariant with respect to the almost complex structure and the Hamiltonian isotopy bringing L to $\phi_H(L)$. We are currently working to prove these invariance properties.

3. Future collaboration with the host institution

Future collaboration with the host institution is extremely likely, as we still have much work to do on the spectral invariants for displaceable Lagrangian submanifolds. We must also finish preparing the article on the energy-capacity inequalities for Lagrangian submanifolds with $\omega|_{\pi_2(M,L)} = \mu|_{\pi_2(M,L)} = 0$, and subsets of $\mathbb{R}^{n+k}$ properly embedded in some $2n$-dimensional manifold $U \subset \mathbb{R}^{2n}$.

4. Projected publications resulting from this grant

The work done during the period of the grant will primarily strengthen a paper already in progress, and during this visit we were able to get this paper most of the way toward being ready to submit for publication. It is also possible that the spectral invariant for displaceable Lagrangians will become the subject of another publication.

References

1. Helmut Hofer and Eduard Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhauser, 1994.
2. Rémi Leclercq, *Spectral Invariants in Lagrangian Floer Theory*, Journal of Modern Dynamics **2** (2008), no. 2, 249–286.