Short Visit Grant Report:
Fermionic entanglement in small rings

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Fermions are essential constituents of matter. Due to the Pauli exclusion principle fermions have to be described by antisymmetric states: any interchange of two fermions results in sign change of the wave function. Due to this deepest property of Nature quantum state of two fermions is always entangled, i.e. the reduced state (with respect to one of fermions) is always mixed. There are however among such entangled states these which can be 'more entangled' then the other. Such type of reasoning leads to the notion(s) (there is more than one) of *fermionic entanglement*. Ambiguities in defining what fermionic entanglement really means are related to the lack of a natural direct product structure of the state space of multi–fermion system analogous to the tensor product of spaces of distinguishable particles. In one of appealing attempts, which is going to be adapted in our present work, the Slater rank resembling Schmidt rank is proposed which indicates a number of terms in the Schmidt decomposition of the state of the composite system. Slater rank of fermionic states counts non–zero Slater determinants in the expansion of the state in a certain basis and therefore allows to distinguish 'more entangled' states from 'less entangled' states.

During my visit in Augsburg, we have studied translationally invariant fermionic configuration: a ring of N sites accomodating $\mathcal{N} = 2$ fermions. A general state vector in the $2N$-dimensional one-particle space can written in the form

$$|\psi\rangle = \sum_{i,j=1}^{N} \sum_{\sigma,\sigma'=\uparrow,\downarrow} w_{i\sigma,j\sigma'} \hat{c}^{\dagger}_{i,\sigma} \hat{c}^{\dagger}_{j,\sigma'} |0\rangle, \tag{1}$$

where $\hat{c}^{\dagger}_{i,\sigma}$ creates a fermion with spin σ in a single-particle state i. The coefficients $w_{i\sigma,j\sigma'}$ form the antisymmetric matrix W. The rank of the matrix W is the criterion to judge entanglement or separability: if the rank$W = 2$ then the state is separable; if the rank$W > 2$ then the state is entangled. Equivalently, the state $|\psi\rangle$ in Eq. (1) can be expanded into a basis of elementary Slater determinants with a minimum number r of non-vanishing terms. The number r is called the fermionic Slater rank of the state $|\psi\rangle$. States with Slater rank 1 are natural analogues of product states and therefore are nonentangled. If Slater rank is greater than 1, the state is entangled. The entanglement criterion for states of two fermions can be formulated in terms of the linear entropy S of the reduced density matrix ρ of the single fermion. It is constructed from the pure state $\varrho = |\psi\rangle\langle\psi|$ of the two-particle system by tracing over the states belonging to the second particle, i.e., $\rho = \mathrm{Tr}_2 \varrho$ (taking a trace over the states belonging to the first particle gives the same result). The linear entropy of the one-particle state ρ is defined as

$$S = 1 - \mathrm{Tr}[\rho^2] = 1 - \sum_{i,j=1}^{N} \sum_{\sigma,\sigma'=\uparrow,\downarrow} \Big|\sum_{l=1}^{N} \sum_{\sigma''=\uparrow,\downarrow} w_{i\sigma,l\sigma''} w^{*}_{l\sigma'' j\sigma'}\Big|^2 \tag{2}$$

and can be calculated directly from the matrix W. On the other hand, the matrix elements of ρ can be computed as

$$\rho_{i\sigma,j\sigma'} = \langle\psi|\hat{c}^{\dagger}_{j,\sigma'} \hat{c}_{i,\sigma}|\psi\rangle/\mathcal{N}, \qquad \mathcal{N} = \sum_{i=1}^{N} \sum_{\sigma=\uparrow,\downarrow} \langle\psi|\hat{c}^{\dagger}_{i,\sigma} \hat{c}_{i,\sigma}|\psi\rangle, \tag{3}$$

where $\hat{c}_{i,\sigma}$ is the annihilation operator of the fermion.

Certain symmetries of the state $|\psi\rangle$ may lead to a block–diagonal structure of ρ and, consequently, to much simpler form of Eq. (2). Interacting quantum systems in the thermodynamic limit ($\mathcal{N} \to \infty$) frequently show a spontaneous symmetry breaking when the symmetry of the ground state is lower that the symmetry of the Hamiltonian. However, the problem of the fermionic entanglement is usually addressed for $|\psi\rangle$ being the ground state of a *finite* quantum system, when this type of symmetry breaking is rare or at least non–generic. Hence the case when $|\psi\rangle$ has the same symmetry as the Hamiltonian is physically most relevant. For the common case of translationally invariant systems, the single–particle density–matrix is block-diagonal in the momentum representation, i.e.,

$$\langle\psi|\hat{c}^{\dagger}_{\vec{k},\sigma} \hat{c}_{\vec{k}',\sigma'}|\psi\rangle = \delta_{\vec{k},\vec{k}'} n^{\vec{k}}_{\sigma,\sigma'} \tag{4}$$

for wave–vectors $\vec{k}$ and $\vec{k}'$. Further on, we consider a generic case when the z-component the total spin is a well defined (conserved). Then

$$\langle\psi|\hat{c}^{\dagger}_{\vec{k},\sigma} \hat{c}_{\vec{k}',\sigma'}|\psi\rangle = \delta_{\vec{k},\vec{k}'} \delta_{\sigma,\sigma'} n_{\vec{k},\sigma} \tag{5}$$

and the linear entropy takes the form

$$S = 1 - \frac{1}{\mathcal{N}^2} \sum_{\vec{k}} \sum_{\sigma} n^2_{\vec{k},\sigma}. \tag{6}$$

This simple reasoning shows that for the common case of translationally invariant system with conserved z–component of the total spin, the fermions entanglement is a function of the momentum (and/or spin) distribution $n_{\vec{k},\sigma}$. **The latter quantity has been measured for many condensed–matted and cold–atoms systems**. It is also quite evident that the above reasoning holds for arbitrary $\mathcal{N} \geq 2$. Then, the linear entropy quantifies the entanglement between a single fermion and its surrounding consisting of $\mathcal{N} - 1$ fermions.

We have assumed that the dynamics of fermions moving in the ring is determined by the Hamiltonian

$$H = t \sum_{i=1}^{N} \sum_{\sigma=\uparrow,\downarrow} \left(\hat{c}^{\dagger}_{i+1,\sigma} \hat{c}_{i,\sigma} + h.c \right) + U \sum_{i=1}^{N} \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow} + C \sum_{i \neq j}^{N} \sum_{\sigma,\chi=\uparrow,\downarrow} \frac{1}{|i-j|} \hat{n}_{i,\sigma} \hat{n}_{j,\chi}, \tag{7}$$

i.e. we have considered interaction between two fermions by including both local (intra–site) and non–local (inter–site) interactions. The operator $\hat{n}_{i,\sigma} = \hat{c}^{\dagger}_{i,\sigma} \hat{c}_{i,\sigma}$, the parameter t describes hopping, U is on–site Coulomb repulsion and C characterizes the inter–site Coulomb–type coupling.

For a maximum entangled state the linear entropy is

$$S_{max} \;=\; 1 - \frac{1}{2N}. \tag{8}$$

Because it depends on dimension N, we have investigated the rescalled linear entropy

$$\mathbb{S} = \frac{S - \frac{1}{2}}{S_{max} - \frac{1}{2}} \in [0, 1] \tag{9}$$

We have started with a simplest and illustrative example: a dimer $N = 2$. The two–fermion ground state reads

$$|E_0\rangle = \left[a_1 \left(\hat{c}^{\dagger}_{1,\uparrow} \hat{c}^{\dagger}_{1,\downarrow} + \hat{c}^{\dagger}_{2,\uparrow} \hat{c}^{\dagger}_{2,\downarrow} \right) + a_2 \left(\hat{c}^{\dagger}_{1,\uparrow} \hat{c}^{\dagger}_{2,\downarrow} - \hat{c}^{\dagger}_{1,\downarrow} \hat{c}^{\dagger}_{2,\uparrow} \right) \right] |0\rangle, \tag{10}$$

where

$$a_{1/2} = \frac{1}{2} \sqrt{1 \mp \frac{U - C}{\sqrt{(U-C)^2 + 16t^2}}}. \tag{11}$$

For this ground state, we have analytically calculated the rescaled linear entropy. It takes the form

$$\mathbb{S} = \frac{\delta^2}{1 + \delta^2}, \quad \delta = \frac{U - C}{4t}. \tag{12}$$

We observe that for $N = 2$ the ground state entanglement depends only on the rescaled difference $|\delta|$ between local U and non–local C interaction amplitudes. This feature is not generic but rather specific for this simplest configuration. For the same strength of intra-site and inter-site interactions, i.e. when $\delta = 0$ (independently how large or small are amplitudes U and C), the entropy is minimal, $\mathbb{S} = 0$, and entanglement of the ground state is minimal. The second limiting case is when one type of the interaction (no matter which) dominates, i.e. $|\delta| >> 1$. In this regime the entropy approaches the maximal value $\mathbb{S} = 1$ and entanglement of the ground state (10) is maximal. It is interesting that the type of interaction is not important but only relation between U/t and C/t plays a significant role.

Entanglement of a ground state for systems of greater number of sites $N > 2$ will be analyzed soon. We have performed numerical calculations. However, we need time to finalize deeper analysis. The preliminary conclusion is: an information shared by two fermions occupying remote sites is smaller for larger distances. For non–local interactions included the situation becomes even more complicated. There are certain circumstances (e.g. $U = C$ for $N = 2$) when local and non–local interaction between fermions acts destructively for their entanglement. It is also the case for $N = 4$ whereas seems to be absent for larger systems at least in the tailored range of parameters.

Main results:

- Translationally invariant systems allow to relate fermionic entanglement to measureable quantities: Eq.(6)
- Local interaction U is better for controlling entanglement than non–local interaction C in steady states.