Scientific report

1. Purpose of the visit. A participation in the ESI workshop *Large cardinals and descriptive set theory*, taking place in Vienna in June 14–27, 2009.

2. Description of the work carried out during the visit. During the workshop, I advanced my investigations in several directions:

2.1. combinatorics without the axiom of choice,

2.2. the algebra of ultrafilters,

2.3. infinitary logic,

2.4. infinitary analogs of descriptive set theory,

and some others. Conversations with some of my colleagues were useful for me.

3. Description of the main results obtained.

3.1. Working in ZF alone, i.e. without AC, I give an explicit evaluation of the Hartogs function $\aleph$ and its surjective analog $\aleph^*$ at a set via the values of $\aleph$ and $\aleph^*$ at a covering of this set and elements of the covering. In particular, I prove that any infinite well-ordered successor cardinal λ^+ cannot be covered by λ sets each of cardinality less than λ. Contrasting with many "negative" results (due by Jech, Gitik, and others), this shows that the behavior of cardinals without AC is not "arbitrarily bad".

3.2. I show that well-known ZFC facts about quasi-disjoint families can be recovered in ZF alone. The proof uses results of 3.1. The only case remaining unclear for me is when the cardinality of a given family is a successor cardinal of countable cofinality; in this case I've got a bit weaker result than ZFC proves about singular cardinals, and at the moment I don't know whether it is possible to get the same as in ZFC.

3.3. A standard result says that any compact left topological semigroup has an idempotent; this allows to use idempotent ultrafilters to obtain a lot of theorems in number theory, algebra, and dynamics. I extend this result by proving that any compact left topological left semiring has a common, i.e. additive and multiplicative simultaneously, idempotent. I prove also similar results for more general universal algebras. As an application, I partially answer a question about the algebra of $\beta\mathbb{N}$, the space of ultrafilters over natural numbers.

3.4. Mycielski asked whether all subsets of the space $\kappa^{\mathrm{cf}\,\kappa}$ with its lexicographic order topology that are in $L[\kappa^{\mathrm{cf}\,\kappa}]$ may have the properties that are immediate analogs of the Baire property and the perfect set property, for all cardinals κ. I show that then $V \neq L[S]$ for any set S and V contains a certain portion of large cardinals. Mycielski asked also more generally about appropriate descriptive set theories for these spaces. I show also that, under some assumptions on κ, many concepts and results of classical descriptive set theory have immediate analogs for the space $\kappa^{\mathrm{cf}\,\kappa}$.

3.5. The completeness theorem, like most of theorems of the usual (finitary) logic $\mathcal{L}_{\omega,\omega}$, fails in many infinitary logics $\mathcal{L}_{\kappa,\lambda}$. E.g. Scott's undefinability theorem states that $\mathcal{L}_{\omega_1,\omega_1}$ refutes completeness, even in a strong sense: the set of valid formulas is not definable in the set of all formulas, unlike the set of provable formulas. The same holds for any successor cardinal instead of ω_1. The natural question is to determine which logics $\mathcal{L}_{\kappa,\lambda}$ satisfy completeness. I show that in fact completeness is equivalent to compactness; any consistent theory in $\mathcal{L}_{\kappa,\lambda}$ has a model if and only if κ is strongly compact, and any consistent theory in $\mathcal{L}_{\kappa,\lambda}$ using at most κ non-logical symbols has a model if and only if κ is weakly compact.

4. Projected publications resulting from the grant. The following of my papers are finalized:

4.1. *Notes on cardinals without the axiom of choice*, to appear.

4.2. *Quasi-disjoint families without the axiom of choice.*

4.3. *Common idempotents in compact left topological algebras.*

4.4. *Completeness in infinitary logic*, to appear.

The following works are in progress:

4.5. *On Hindman sets.*

4.6. *On a question of Mycielski.*

All of these papers (and possibly some of others) will contain an acknowledgment of the partial support by the INFTY grant .