ESF Short Visit Grant
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Does higher-dimensional forcing prove the consistency of $\mathfrak{b} = \kappa \wedge \mathfrak{s} = \mathfrak{a} = \kappa^{++}$?

Scientific Report

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Purpose of the visit

The structure of the real numbers has been studied ever since the discovery of transfinite numbers by Cantor. Among the things studied are the cardinal characteristics $\mathfrak{b}$, $\mathfrak{s}$ and $\mathfrak{a}$. Assume that the reals are represented as functions $f : \omega \to \omega$. A function $f : \omega \to \omega$ is said to be dominated by $g : \omega \to \omega$ if $f(n) \geq g(n)$ for all but finitely many $n \in \omega$. A family B of functions $f : \omega \to \omega$ is unbounded if there is no $g : \omega \to \omega$ which dominates all $f \in B$. The bounding number is defined by $\mathfrak{b} = \{card(B) \mid B$ is unbounded$\}$. Similarly, the splitting number $\mathfrak{s}$ is the size of a minimal splitting family and the almost disjointness number $\mathfrak{a}$ is the minimal size of a maximal almost disjoint family.

S. Shelah [10] proved by a countable support iteration of proper forcing that $\mathfrak{b} = \omega_1 \wedge \mathfrak{s} = \mathfrak{a} = \omega_2$ is consistent. J. Brendle [2] showed the consistency of $\mathfrak{b} = \omega_1 \wedge \mathfrak{s} = \kappa$ for regular $\kappa > \omega$. The main result of V. Fischer's PhD thesis [4, 5] is the consistency of $\mathfrak{b} = \kappa \wedge \mathfrak{s} = \mathfrak{a} = \kappa^+$ for regular $\kappa > \omega$. She uses a thinned out version $Q(C)$ of Shelah's forcing notion, which is ccc, and iterates it with finite support. Her main lemma reads as follows:

Let $\kappa > \omega$ be regular, $cov(meager) = \kappa$, H be an unbounded $<^*$-dirceted family, $card(H) = \kappa$, $\forall\lambda < \kappa(2^\lambda \leq \kappa)$. Then there is a centered family C such that $card(C) = \kappa$, $Q(C) \Vdash H$ is unbounded and $Q(C)$ adds a real not split by the ground model reals.

She iterates the forcing $Q(C)$ κ^+-many times. Moreover, in every step she adds κ Cohen reals to keep $cov(meager) = \kappa$ and performs a Hechler forcing to kill small unbounded families. However, it is impossible to iterate it more often, because then the assumptions of the lemma will not be satisfied anymore.

To overcome this difficulty, I propose to use a two-dimensional forcing instead of a linear iteration. I developed a theory of such forcings in [7, 8, 9]. In our case, it would mean to construct the forcing along a simplified $(\kappa^+, 1)$-morass. This is possible if the iteration has certain properties. In particular, it is necessary that there are complete embeddings between certain subsets of the forcings. The iteration of Cohen forcing has this property for trivial reasons. Also Hechler forcing is known to have it. Finally, I expect that the Q(C) can be thinned out along the morass, so that it obtains the necessary property. This construction is related to the generic construction of C in [6]. To check this together with Vera Fischer, I visited the Kurt Gödel Research Center for Mathematical Logic in Vienna from May 17 until May 23, 2009.

Description of the work carried out during the visit

Hechler forcing $\mathbb{H}$ has the following property:

Let $\mathbb{P}, \mathbb{Q}$ be forcings and $f : \mathbb{P} \to \mathbb{Q}$ be a complete embedding. Let $q \in \mathbb{Q}$ and $p \in \mathbb{P}$ be a reduction of

q with respect to f. Then f extends in a natural way to a complete embedding $\tilde{f} : \mathbb{P} * \mathbb{H} \to \mathbb{Q} * \mathbb{H}$. That is, if $q \Vdash (s, \dot{h}) \in \mathbb{H}$, then we can find a $\mathbb{P}$-name $\dot{g}$ such that $(p, (s, \dot{g})) \in \mathbb{P} * \mathbb{H}$ is a reduction of $(q, (s, \dot{h})) \in \mathbb{Q} * \mathbb{H}$.

Assume that $\delta < \lambda$ are ordinals and $\mathbb{P}$ is an iteration of Hechler forcing of length δ while $\mathbb{Q}$ is one of length λ. Let $\pi : \delta \to \lambda$ be order-preserving. Then we can use the above property to find a complete embedding $f : \mathbb{P} \to \mathbb{Q}$ such that $supp(q) = \pi[supp(p)]$ if $f(p) = q$.

The aim is to do something similar for the iteration of Vera Fischer's PhD thesis and for morass-maps π. As a first step, we looked at an iteration of the forcings $\mathbb{C}(\kappa) * Q(C)$ where $\mathbb{C}(\kappa)$ is the forcing which adds κ-many Cohen reals and $Q(C)$ is the forcing with logarithmic measures which was explained in the previous section. The main obstacle in proving a property like the above one for this iteration is that the centered family C has a complicated recursive definition. Hence there is a priori no reason why some $Q(C_1)$ of the iteration should in any way be related to a $Q(C_2)$ which appears at a later stage of the iteration.

Therefore, we formulated V. Fischer's consistency proof of $\mathfrak{b} = \kappa \wedge \mathfrak{s} = \kappa^+$ as a matrix construction like in Blass' and Shelah's work [1] on ultrafilters. To do so, one has to replace the main lemma of [1] by an appropriate version for the $Q(C)$. This is the following

Main lemma. Let $M \subseteq N$ be models of ZFC such that $C \in M$ is a maximal centered family of pure conditions in M. Let $g \in {}^{\omega}\omega \cap N$ be an unbounded real over M. Then there exists a maximal centered family C' in $N^{\mathbb{C}(\kappa)}$ where $\kappa = (2^\omega)^N$ such that

1. $C \subseteq C'$
2. Every maximal antichain of $Q(C)$ from M remains maximal in $Q(C')$
3. For all $Q(C)$-names $\dot{f}$ in M for reals, $\Vdash^N_{Q(C')}$ "$\check{g} \not\leq^* \dot{f}$".

Then we proceed exactly like in [1] and obtain the consistency of $\mathfrak{b} = \kappa \wedge \mathfrak{s} = \kappa^+$. Note, that because of the condition $\kappa = (2^\omega)^N$ in the main lemma, we can still not get $\mathfrak{b} = \kappa \wedge \mathfrak{s} = \kappa^{++}$. However, as J. Brendle pointed out to us, this is possible if we replace $Q(C)$ by the usual Mathias forcing over ultrafilters (see also [2]).

The next question is, if the matrix construction is uniform enough to construct complete embeddings as described in the beginning. We found out that this can be done for Mathias forcing over filters $\mathbb{M}_U$. More precisely, we are able to construct along a simplified $(\kappa^+, 1)$-morass a matrix iteration of height κ and length κ^{++} of Mathias forcings over filters $\langle \dot{U}^\gamma_\eta \mid \gamma \leq \kappa, \eta < \kappa^{++} \rangle$. As before, we fix a sequence $\langle c_\gamma \mid \gamma < \kappa \rangle$ of κ-many Cohen reals. We set $V_\gamma := V[\{c_\delta \mid \delta < \gamma\}]$. For each $\gamma \leq \kappa$ we will have a finite support iteration $\langle \mathbb{P}^\gamma_\eta, \dot{\mathbb{Q}}^\gamma_\eta \mid \eta < \kappa^{++} \rangle$ such that for all η,

$$\Vdash_{\mathbb{P}^\gamma_\eta} \dot{\mathbb{Q}}^\gamma_\eta \simeq \mathbb{M}_{\dot{U}^\gamma_\eta}$$

and

1. $\mathbb{P}_\eta$ forces that $\dot{U}^\lambda_\eta$ is for all $\lambda \in Lim$, $\lambda \leq \kappa$ the increasing union of the $\dot{U}^\gamma_\eta$, $\gamma < \lambda$,
2. if $\eta \leq \kappa^{++}$ and $\gamma < \delta \leq \kappa$, and A is a maximal antichain of $\mathbb{P}^\gamma_\eta$ in V_γ, then A is a maximal antichain of $\mathbb{P}^\delta_\eta$ in V_δ,
3. if $\eta \leq \kappa^{++}$ and $\gamma < \delta \leq \kappa$, then in V_δ, whenever $\dot{f} \in V_\gamma$ is a $\mathbb{P}^\gamma_\eta$-name for a real, then $\Vdash_{\mathbb{P}^\delta_\eta} c_\gamma \not\leq^* \dot{f}$.

Let $\mathfrak{M}$ be a simplified $(\kappa^+, 1)$-morass. Let as usual $\prec$ be its tree relation and $\pi_{st} : \nu(s) + 1 \to \nu(t) + 1$ be the associated system of maps $\langle \pi_{st} \mid s \prec t \rangle$. We define $\langle \dot{U}^\gamma_\eta \mid \gamma \leq \kappa, \eta < \kappa^{++} \rangle$ by induction over the the levels of the morass which we enumerate by $\beta \leq \kappa^+$. Moreover, we construct a system $\langle \sigma^\gamma_{st} \mid \gamma \leq \kappa, s \prec t \rangle$ of complete embeddings $\sigma^\gamma_{st} : \mathbb{P}^\gamma_{\nu(s)+1} \to \mathbb{P}^\gamma_{\nu(t)+1}$, $\sigma^\gamma_{st} \in V_\gamma$, such that, if $\pi_{st}(\nu) = \tau$ and $s' = \langle \alpha(s), \nu \rangle$, $t' = \langle \alpha(t), \tau \rangle$, then σ^γ_{st} extends $\sigma^\gamma_{s't'}$ where we view $\mathbb{P}^\gamma_{\nu+1}$ in the obvious way as a subset of $\mathbb{P}^\gamma_{\nu(s)+1}$. Set

$$\sigma^\gamma_{<st} = \bigcup \{\sigma^\gamma_{s't'} \mid \nu < \nu(s), s' = \langle \alpha(s), \nu \rangle, t' = \langle \alpha(t), \tau \rangle, \pi_{st}(\nu) = \tau\}.$$

Then $\sigma^\gamma_{<st} : \mathbb{P}^\gamma_{\nu(s)} \to \mathbb{P}^\gamma_{\nu(t)}$ is a complete embedding.

We don't know yet if such a construction can be carried out for the forcings $Q(C)$ instead of $\mathbb{M}_U$.

To get the consistency of $\mathfrak{b} = \kappa \wedge \mathfrak{s} = \mathfrak{a} = \kappa^{++}$ one needs to combine the above construction with a forcing that makes $\mathfrak{a}$ big. Such a forcing $\mathbb{P}(A)$ was described by J. Brendle in [3]. $\mathbb{P}(A)$ is ccc and destroys the mad family A. Since $\mathbb{P}(A) = \mathbb{C}(\kappa) * \mathbb{M}_{\dot{F}}$ where $\dot{F}$ is a $\mathbb{C}(\kappa)$-name for a filter F, all techniques described so far apply. The only problem seems to be to make the book keeping of the iteration compatible with the morass construction. I expect that this can be done with a universal version of a simplified morass or a simplified morass with built in diamond.

Description of the main results being obtained

1. $\mathbb{C}(\kappa) * Q(C)$ can be iterated in a matrix of height κ and length κ^+ like in [1].

2. $\mathbb{C}(\kappa) * Q(C)$ is not necessary to obtain the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \lambda$ for arbitrary regular κ, λ. This can be done by a matrix iteration of $\mathbb{C}(\kappa) * \mathbb{M}_U$ of height κ and length λ. Nevertheless, it would be interesting if a matrix iteration of $\mathbb{C}(\kappa) * Q(C)$ of height κ and length κ^{++} can be obtained.

3. The matrix iteration of $\mathbb{C}(\kappa) * \mathbb{M}_U$ of height κ and length κ^{++} can be obtained as morass limit.

4. The techniques developed so far also apply to a forcing that destroys mad families, hence $\mathfrak{b} = \kappa \wedge \mathfrak{s} = \mathfrak{a} = \kappa^{++}$ seems possible. However, it is not yet clear how to combine book keeping with the morass approach.

Future collaboration with host institution

Since we haven't proved the consistency of $\mathfrak{b} = \kappa \wedge \mathfrak{s} = \mathfrak{a} = \kappa^{++}$ yet, we will continue to do so. It is necessary to solve the following question:

1. Can the book keeping of iterated forcing be combined with the morass construction?

Of independent interest is the following question, which should be easily answered by a close analysis of the argument used for $\mathbb{C}(\kappa) * \mathbb{M}_U$:

2. Can a matrix iteration of $\mathbb{C}(\kappa) * Q(C)$ of height κ and length κ^{++} be obtained?

To solve 1 and 2, further visits might be necessary.

Projected publications/articles resulting or to result from the grant

The obtained results are interesting in their own right. However, it seems premature to publish them, because we haven't proved the consistency of $\mathfrak{b} = \kappa \wedge \mathfrak{s} = \mathfrak{a} = \kappa^{++}$.

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