Scientific Report:
Short Visit to Barcelona

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Purpose of the Visit

The purpose of this visit was to begin working with Professors Joan Bagaria and Carles Casacuberta, both of the University of Barcelona, on problems at the interface between set theory and algebraic topology.

Description of Work Carried Out

Vopěnka's Principle is a large cardinal axiom which has been extensively studied by category theorists, and which has recently found applications to algebraic topology and related aspects of category theory. In this visit we undertook to study Vopěnka's Principle and its potential applications further.

A variant of Vopěnka's Principle known as *Weak Vopěnka's Principle* is known to be equivalent to many statements of interest in category theory, and so was a natural choice as the focus of our work. It is known that Weak Vopěnka's Principle follows from Vopěnka's Principle, and implies the existence of a proper class of measurable cardinals. However, this leaves a large gap of consistency strengths, which Professor Bagaria and I worked to try to narrow.

We also discussed with Professor Casacuberta some applications of Vopěnka's Principle on which he had been working, and made preliminary steps in planning a proposed longer visit for October.

Description of Main Results Obtained

Whilst Professor Bagaria and I were not yet able to improve the consistency strength bounds known for Weak Vopěnka's Principle, we made some progress in translating large cardinal concepts into the appropriate category-theoretic framework. In particular, we developed a formulation of strongly compact and supercompact cardinals in a category-theoretic context.

We were aiming to emulate the proof that the existence of only a set of measurable cardinals implies the negation of Weak Vopěnka's Principle. The

first step of this proof is the following observation. For a given set X and cardinal κ, let the *canonical diagram for* X be the diagram with, as objects, a copy λ_f of $\lambda < \kappa$ for each function $f : X \to \lambda$, and as morphisms, the functions g between objects λ_f and $\mu_{g \circ f}$. Then (forgetting about X now), elements of the limit of this diagram can be thought of as choosing a piece from every partition of X into less than κ many pieces in a coherent way, and thus correspond exactly to κ-complete ultrafilters on X. If κ is greater than all measurable cardinals, then the only such ultrafilters are the principal ones, and X can be naturally identified with the limit of its own canonical diagram.

To extend this technique to strongly compact or supercompact cardinals, we considered the canonical diagram for $\mathcal{P}_\kappa(X)$. We discovered that by adding appropriately chosen functions to the diagram we could restrict elements of the limit to be fine ultrafilters or fine normal ultrafilters, as witness $|X|$-strong compactness and $|X|$-supercompactness respectively. However, we noted that principal unltrafilters on $\mathcal{P}_\kappa(X)$ are *not* fine, so extending the analogy with the measurable case in order to get supercompact cardinals from Weak Vopěnka's Principle seemed problematic.

In our discussion with Casacuberta, Bagaria and I were also able to point out an instance where assuming Vopěnka's Principle was massive overkill, with "ORD is Mahlo" being sufficient for the case at hand.

Future Collaboration

We are proposing that I return to Barcelona for the month of October to continue this work. It is further planned that Jirí Rosický and Adrian Mathias, who have also previously worked with Bagaria and Casacuberta in this area, be present at that time if possible.

Publications

No publications have arisen from this short visit. However, we are hopeful for the future output of the collaboration.