Technical Report for INFTY Short Visit - 3548.

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1 Introduction.

The first purpose of this visit was to begin a cooperation with P. Borodulin-Nadzieja on the following two topics: Nowhere weak distributivity of ccc Boolean algebras (what we have learnt is described in section 2) and the fragmentation of measures in topological spaces (what we learnt is described in section 3). The second purpose of this visit was to gain feedback on my PhD studies thus far.

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2 Nowhere Weakly Distributive Boolean Algebras

A complete Boolean algebra $\mathfrak{B}$ is **weakly distributive** if and only if for every $(a_{m,n})_{m,n\in\omega} \subseteq \mathfrak{B}$ with $a_{m,n} \leq a_{m,n+1}$ and $\sum_n a_{m,n} = 1$ we have

$$\sum_{f\in{}^{\omega}\omega}\prod_{m\in\omega} a_{m,f(m)} = 1.$$

The notion of weak distributivity was introduced by von Neumann in [7], who asked whether or not every complete ccc weakly distributive Boolean algebra carries a positive σ-additive measure. In [8], Talagrand constructed a complete ccc weakly distributive Boolean algebra that does not carry such a measure.

We call $\mathfrak{B}$ **nowhere weakly distributive** if and only if for every $a \in \mathfrak{B}^+$, $\mathfrak{B} \upharpoonright a$ is not weakly distributive. In section 2.1 we discuss what nowhere weak distributivity means for forcing extentions of $\mathfrak{B}$. In section 2.2 we consider what nowhere weak distributivity means with respect to the *sequential topology* on $\mathfrak{B}$.

2.1 Generic Reals

Recall that a subset $F \subseteq {}^{\omega}\omega$ is called **dominating** if and only if $(\forall f \in {}^{\omega}\omega)(\exists g \in F)(g \geq^* f)$. The following proposition is well known and is not difficult to prove.

Proposition 1 *Let $\mathfrak{B}$ be a complete ccc Boolean algebra. Then $\mathfrak{B}$ is nowhere weakly distributive if and only if*

$$\mathbb{1} \Vdash {}^{\omega}\omega \cap M \text{ is not a dominating family.} \tag{2.1}$$

Example 1. It is easy to see from proposition 2.1 that if $\mathfrak{B}$ is ccc and adds a Cohen real then $\mathfrak{B}$ is nowhere weakly distributive.■

The converse to this example is almost true. Recall that a forcing notion $\mathbb{P} = (P, \leq)$ is **Suslin** if and only if P, $\leq$ and the incompatibility relation in $\mathbb{P}$ are expressible as Σ^1_1 relations in ω^ω (see [3, page 168]).

Theorem 1 ([3, page 190]) *If $\mathfrak{B}$ is ccc Suslin (as a forcing notion) then $\mathfrak{B}$ adds a Cohen real if and only if it is nowhere weakly distributive.*

The following example however shows that the converse of this outside of Suslin notions will not be true in ZFC.

Example 2. Let $\mathcal{U}$ be a Ramsey ultrafilter on ω. Recall that the Mathias forcing $\mathbb{M}$ relativised to $\mathcal{U}$ is defined as follows:

- $(s, A) \in \mathbb{M}$ if and only if $s \in {}^{<\omega}\omega$, $A \in \mathcal{U}$ and $\max s < \min A$.
- $(s, A) \leq (t, B)$ if and only if $s \supseteq t$, $A \subseteq B$ and $ran(s \setminus t) \subseteq B$.

This example has the following properties:

- $\mathbb{M}$ is ccc (because for any $(s, A), (s, B) \in \mathbb{M}$ we have $(s, A \cap B) \leq (s, A), (s, B)$ and $|{}^{<\omega}\omega| = \aleph_0$),
- $\mathbb{M} \Vdash (\exists f)(\forall g \in V)(f \not\leq^* g)$.
- $\mathbb{M}$ satisfies the *Laver property* and therefore it cannot add any Cohen reals (nor can it add any random reals).

Thus the Boolean completion of $\mathbb{M}$ is a ccc nowhere weakly distributive Boolean algebra that does not add a Cohen real.[1] ■

2.2 Sequential Topology

Let $\mathfrak{B}$ be a complete ccc Boolean algebra. Recall the following topology (**sequential topology**) on $\mathfrak{B}$: A set $C \subseteq \mathfrak{B}$ is closed if and only if for every sequence $(a_n)_{n\in\omega} \subseteq C$ either $\sum_{n\in\omega}\prod_{k\geq n} a_k \neq \prod_{n\in\omega}\sum_{k\geq n} a_k$ or $\sum_{n\in\omega}\prod_{k\geq n} a_k \in C$. A **submeasure** is a function $\mu : \mathfrak{B} \to \mathbb{R}_{\geq 0}$ that satisfies the following three conditions:

- $\mu(0) = 0$,
- if $a \leq b$ then $\mu(a) \leq \mu(b)$,
- $\mu(a \cup b) \leq \mu(a) + \mu(b)$.

We say that μ is **strictly positive** if $\mu(a)$ is non-zero whenever a is. We call μ a **Maharam submeasure** if and only if $x_n \to_s x$ implies that $\mu(x_n) \to \mu(x)$.

Definition 1 *Say that $\mathfrak{B}$ is* **anti-Housorff** *if and only if every non-empty open set in the sequential topology of $\mathfrak{B}$ is topologically dense.*

We will use the following result.

Theorem 2 ([2]) *If $\mathfrak{B}$ is a complete ccc Boolean algebra then there exists an $m \in \mathfrak{B}$ (possibly 0) such that*

- *$\mathfrak{B} \restriction m$ carries a strictly positive Maharam submeasure*
- *$\mathfrak{B} \restriction (1 \setminus m)$ is anti-Housdorff.*

In [1, page 255] it is shown that $\mathfrak{B}$ is anti-Housdorff if $\mathfrak{B}$ is nowhere weakly distributive. The following example shows that the converse of this fact is not true under the negation of Suslin's Hypothesis.

Example 3. A **Suslin algebra** is a complete atomless ccc Boolean algebra that satisfies the following (ω,ω)**-distributivity** law: *For every double sequence $(a^n_m)_{m,n}$*

$$\bigcap_{n\in\omega}\bigcup_{m\in\omega} a^n_m = \bigcup_{f\in\omega^\omega}\bigcap_{n} a^n_{f(n)}.$$

[1] The proofs of these facts are given in [3, page 363] (modulo the restriction to a Ramsey ultrafilter which requires a straight forward modification).

The existence of a Suslin algebra is equivalent to the existence of a Suslin tree (see [5, page 229]). A Suslin algebra cannot carry a strictly positive Maharam submeasure (see [1, page 246]). Clearly every principal ideal of a Suslin algebra is also a Suslin algebra and thus in theorem 2 we must have $m = 0$. Since (ω, ω)-distributivity is easily seen to imply weak distributivity a Suslin algebra provides the required counter example.■

For the other direction we have the following.

Proposition 2 *Under Todorcevic's P-ideal dichotomy every complete ccc anti-Housdorff Boolean algebra is nowhere weakly distributive.*

Proof. If $\mathfrak{B}$ is anti-Housdorff then for every $a \in \mathfrak{B}^+$, $\mathfrak{B} \upharpoonright a$ is anti-Housdorff and therefore cannot carry a strictly positive Maharam submeasure (if it did then the sequential topology would be metrisable (see [6, page 157]) and therefore Housdorff). Since under PID every complete ccc weakly distributive Boolean algebra carries a strictly positive Maharam submeasure (see [1, page 262]), $\mathfrak{B} \upharpoonright a$ cannot be weakly distributive.■

3 Fragmentation of Measures

For a compact space K let $P(K)$ denote the space of probability measures on K with the weak*-topology. One can distinguish different levels of complexity of $P(K)$. Let $S_0(K)$ be the family of measures of finite support. For $\alpha < \omega_1$ denote (after [4]) by $S_{\alpha+1}(K)$ the set of measures being limits of measures from $S_\alpha(K)$. For limit δ let $S_\delta(K)$ be the union of all $S_\alpha(K)$'s for $\alpha < \delta$. Just before the visit we learned of the following result.

Theorem 3 (A. Aviles, G. Plebanek & J. Rodriguez) *Under the continuum hypothesis there exists a compact space K such that $S_1(K) \subsetneq S_{\omega_1}(K) = P(K)$.*

Towards a similar seperation without using the CH we have the following contruction, proposed by P. Borodulin-Nadzieja.

Definition 2 *For $B \in [\omega]^\omega$ and $A \subseteq \omega$ let*

$$d_B(A) = \lim_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}(A),$$

should this limit exists. For $\mathcal{B} = \{B_0, B_1, ...\}$, an infinite partition of ω with each $B_i \in [\omega]^\omega$, and $A \subseteq \omega$ let

$$d_{\mathcal{B}}(A) = \lim_n d_{B_n}(A),$$

should this limit exist.

Construction: Let $\mathcal{B} := \{B_0, B_1, ...\}$ be an infinite partition of ω such that each B_i is infinite. For each $i \in \omega$, let $A_0^i, A_1^i, A_2^i, ...$ be a sequence of partitions of B_i such that

- $(\forall n \in \omega)(|A_n^i| = 2^n)$
- $(\forall n \in \omega)(\forall a \in A_n^i)(\exists b \neq c \in A_{n+1}^i)(a = b \sqcup c)$
- $(\forall n \in \omega)(\forall a \in A_n)(d_{B_i}(a) = \frac{1}{2^n})$.

For each i, let $\mathbb{A}_i$ be the algebra of subsets of B_i generated by $\bigcup_{n \in \omega} A_n^i$. Now let

$$\mathcal{F}_{\mathcal{B}} = \{A \subseteq \omega : (\forall i)(A \cap B_i \in \mathbb{A}_i) \wedge d_{\mathcal{B}}(A) = 1\}.$$

Finally let $\mathbb{B}$ be the algebra of subsets of ω generated by $\mathcal{F}_{\mathcal{B}}$ and $K = Ult(\mathbb{B})$. ■

We have the following.

Proposition 3

1. $\mathcal{F}_{\mathcal{B}} \in K$,

2. $\delta_{\mathcal{F}_{\mathcal{B}}} \in S_2(\omega) \subseteq S_2(K)$.

3. If $X = \{\mathcal{U} \in K : (\exists i)(B_i \in \mathcal{U})\}$ *(in particular* $\omega \subseteq X$*) then* $\mathcal{F}_{\mathcal{B}} \notin S_1(X)$.

■

The hope is that one can extend the algebra $\mathbb{B}$ to a larger algebra (without using CH) in which the measure $\delta_{\mathcal{F}_{\mathcal{B}}}$ is no longer a Dirac measure.

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