Scientific Report: Short Visit to Brussels

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Purpose of the Visit

The purpose of this visit was to begin working with Benedikt Löwe of the University of Amsterdam and Birgit Richter of the University of Hamburg on the set-theoretic structure of the *Bousfield lattice*, an important structure in the area of stable homotopy theory in algebraic topology.

Description of Work Carried Out

The Bousfield lattice is a lattice defined in algebraic topology in a way that encapsulates a lot of important information about stable homotopy theory. In this visit we began a study of it, hoping that a set-theoretic perspective would open new insights to the problems surrounding the lattice. Our focus was the cardinality of the lattice: it is only known to lie between $2^{\aleph_0}$ and $2^{2^{\aleph_0}}$ (inclusive). At a finer scale, there are many interesting sublattices of the Bousfield lattice, including a natural distributive sublattice **DL**, and a way to restrict further to obtain a complete boolean algebra **cBA**. Understanding the cardinalities of these sublattices, and the cardinalities of their complements in each other, was another major goal of the work.

There are of course many settings other than algebraic topology where homological algebra has an important role to play, and in an appropriate general context the Bousfield lattice may be defined for categories other than the stable homotopy category. An example is the derived category $D(\Lambda)$ over a k-algebra Λ for some field k. We also started to dig into what is known in this context; for example, if k is countable then apparently the category $D(\Lambda)$ satisfies a nice "representability" theorem. However, we were unable to get to the bottom of the proof, being unable to consult some references, and even obtaining examples that appear to contradict claims made in the proof. This would be an obvious point to clarify in future work.

Description of Main Results Obtained

Although this was a very short trip at the very beginning of our project, we were able to obtain some interesting results. Most notably, we were able to show that the complement of **cBA** in **DL** is non-empty, a fact that does not appear to have been known before. The proof was an amusing, "non-intuitionistic" proof, giving two candidates (the Bousfield classes denoted $\langle H\mathbb{F}_p \rangle$ and $\langle H\mathbb{F}_p \rangle \vee \langle K(\mathbb{N}) \rangle$), and showing that at least one of them must lie in $\mathbf{DL} \smallsetminus \mathbf{cBA}$. It remains unknown, however, which if either lies in **cBA**. This is a specific example a more general "dichotomy lemma" that we proved.

Lemma 1. *Let X and Y be Bousfield classes in* **DL** *such that $X \wedge Y = \mathbf{0}$, and Z is a Bousfield class less than or equal to both X and Y such that its pseudocomplement AZ is strictly less than $\mathbf{1}$. Then at least one of X, Y, and $X \vee Y$ is not in* **cBA**.

Future Collaboration

We intend to continue this research with a meeting early in 2011, perhaps February.

Publications

Whilst our dichotomy lemma does not seem significant enough to constitute a journal article, the result has been written up and sent to experts in the field for comment.