ESF - Short Visit Grant - Final Report

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The purpose of my visit was better understanding the relationships between logic, Ramsey theory, sofic groups and operator algebras. During my stay, I found that methods from logic can be succesfully applied in the study of sofic and hyperlinear groups. For instance, I observed that sofic groups can be characterized as those (countable discrete) groups that can be embedded in the permutation group of some (or, equivalently, any) infinite hyper-natural number. Analogously, hyperlinear groups can be characterized as those groups that can be embedded in the groups of unitary $\nu \times \nu$ matrices, where ν is some (or, equivalently, any) infinite hyper-natural number. A similar characterization is given by Vladimir Pestov in [P], where it is proved that a (countable discrete) group is sofic (resp. hyperlinear) if and only if it can be embedded in some (or, equivalently, any) ultraproduct of the sequence of finite symmetric groups (resp. finite rank unitary groups). This result motivates the name of universal sofic (resp. hyperlinear) groups for the ultraproducts of the finite symmetric groups (resp finite rank unitary groups). Universal sofic groups have been studied by Simon Thomas in [T], where it is proved that the failure of CH implies that there are $2^{2^{\aleph_0}}$-many non-isomorphic sofic groups. His proof uses some algebraic properties of the finite symmetric groups, and it is not clear if and how it can be modified to get a proof of the analogous statement for hyperlinear groups, namely that the failure of CH implies the existence of $2^{2^{\aleph_0}}$-many nonisomorphic universal hyperlinear groups. I noticed that model theory for metric structures, introduced by Ben Yacoov, Berenstein, Hensov and Usvyatsov in [BYBHU], can be used to get an alternative proof of Thomas' theorem on universal sofic groups, and to get a proof of the analogous statement for hyperlinear groups as well. More precisely, one has to refer to stability theory for metric structures, developed by Farah, Hart, Sherman and Shelah in [FHS2] and [FS] in order to study the number of ultrapowers of C^*-algebras and von Neumann algebras (see [FHS1]). In particular, in those papers it is introduced the so called order property for sequences of metric structures, and it is proved that the failure of CH implies that any sequence with the order property has $2^{2^{\aleph_0}}$-many nonisomorphic metric ultraproducts. The result about universal sofic and hyperlinear groups is deduced from this one showing that the sequences of finite symmetric and unitary groups have the order property.

During my visit, I also gave a talk about these topics at the University of Pisa entitled "*L'ipotesi del continuo ed ultrapotenze di C^*-algebre e algebre di von Neumann*" ("*The continuum hypothesis and ultrapowers of C^*-algebras and von Neumann algebras*"), invited by prof. Mauro di Nasso. An abstract (in Italian) of the talk can be found on the website

http://poisson.phc.unipi.it/~mantova/slap/it/node/29.

The slides (in Italian) from the talk will be soon available on the same website.

Finally, in this period I wrote a survey paper about these topics, that you can find attached in the rest of this document. The paper, entitled "Continuum Logic, Operator Algebras and Sofic and Hyperlinear Groups: A Survey", is aimed to present the results that I summarized in this Scientific Report to a broad public, that does not have necessarily any specific previous knowledge about sofic groups, model theory and operator algebras.

References

[B] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C*-algebras and von Neumann algebras, Operator Algebras and Noncommutative Geometry, III.

[BYBHU] I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model Theory with Applications to Algebra and Analysis, Vol. II (Z. Chatzidakis et al., eds.), Lecture Notes series of the London Math. Society., no. 350, Cambridge University Press, 2008, pp. 315-427

[D] A. Dow, *On ultrapowers of boolean algebras*, Topology Proc. 9 (1984), no. 2, 269-291

[FHS1] I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras I: Stability*, preprint, http://arxiv.org/abs/0908.2790, 2009.

[FHS2] I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras II: Model Theory*, preprint, http://arxiv.org/abs/0908.2790, 2009.

[FS] I. Farah and S. Shelah, *A dichotomy for the number of ultrapowers*, preprint, arXiv:0912.0406v1, 2009

[KP] A. Kwiatkowska, V. Pestov, *An introduction to hyperlinear and sofic groups*, arXiv:0911.4266v2 [math.GR], Lecture notes of the 7-th Appalachian set theory workshop (Cornell University, Ithaca, NY, November 22, 2008), 30 pp., to appear in the volume "Appalachian Set Theory 2006-2009" (James Cummings and Ernest Schimmerling, eds.), the Ontos-Verlag Mathematical Logic Series.

[P] V. Pestov, *Hyperlinear and sofic groups: a brief guide*, Bull. Symbolic Logic 14 (2008), 449-480.

[T] S. Thomas, *On the number of universal sofic groups*, Proceedings of the American Mathematical Society, Volume 138, Number 7, July 2010, Pages 2585–2590

Continuum Logic, Operator Algebras and Sofic and Hyperlinear Groups: A Survey

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Contents

Part I

Operator algebras

Chapter 1

C*-algebras

An (abstract) C*-algebra A is a Banach algebra endowed with an antilinear involution $*$ such that $\|x^*x\| = \|x\|^2$ for every $x \in \mathcal{A}$.

A norm-closed subspace of $B(H)$, where H is a Hilbert space, is a (concrete) C*-algebra. Any abstract C*-algebra is isomorphic to a concrete C*-algebra. A *-homomorphism Φ from a C*-algebra $\mathcal{A}$ to a C*-algebra $\mathcal{A}'$ is an algebraic homomorphism such that $\Phi(x^*) = \Phi(x)^*$ for every $x \in M$. It can be proved that any *-homomorphism is contractive, and hence an injective *-homomorphism is isometric.

An element x of a C*-algebra A is called

- **normal** if $xx^* = x^*x$
- **self-adjoint** if $x = x^*$
- **positive** if $x = y^2$ for some self-adjoint
- **unitary** if $xx^* = x^*x = 1$ (when A has a unit)
- **projection** if $x^* = x = x^2$
- **partial isometry** iff x^*x and xx^* are isometries

If H a Hilbert space, examples of C*-algebras are $B(H)$, the set $\mathcal{K}(H)$ of compact bounded linear operators on H and the quotient $B(H)/\mathcal{K}(H)$, called Calkin algebra. If X is a locally compact Hausdorff space, then the set $C_0(X)$ of continuous complex-valued continuous functions on X vanishing at infinity (namely, functions f such that, $\forall \varepsilon > 0$, $(|f| - \varepsilon) \vee 0$ is compactly supported) with the sup norm and adjunction $f^* = \overline{f}$ is a commutative C*-algebra. Every commutative C*-algebra is isomorphic to a C*-algebra of this kind. An element f of $C_0(X)$ is

- self-adjoint iff $f[X] \subset \mathbb{R}$
- positive iff $f[X] \subset \mathbb{R}_+$

- unitary iff $f[X] \subset \mathbb{S}^1$
- invertible iff $0 \notin f[X]$
- projection iff f is the characteristic function of a connected component of X

A linear functional ϕ on a C*-algebra is called

- **positive** if sends positive elements to positive real numbers
- **state** if it is positive and has norm 1
- **tracial** if $\phi(xy) = \phi(yx)$ for every $x, y \in A$

A positive functional ϕ is automatically bounded, and it is called **faithful** if, for a positive, $\phi(a) = 0$ iff $a = 0$.

Chapter 2

Von Neumann algebras

2.1 Definition and examples of von Neumann algebras

The strong (resp. weak) operator topology on $B(H)$ is defined by the following rule: a net $(T_i)_{i\in I}$ converges to T in the strong (resp. weak) operator topology if and only if for every $x \in H$, the net $(T_i x)_{i\in I}$ converges to Tx in the norm (resp. weak) topology of $B(H)$. The σ-strong (resp. σ-weak) topology on $B(H)$ is defined by the following rule: a net $(T_i)_{i\in I}$ converges to T in the σ-strong or ultrastrong (resp. σ-weak or ultraweak) topology if and only if for every $(x_n)_{n\in\mathbb{N}} \in l^2(H)$, the net $\left(T^i x_n\right)_{n\in\mathbb{N}}$ converges to $(Tx_n)_{n\in\mathbb{N}}$ in the strong (resp. weak) topology of $l^2(H)$.

It is easily seen that the strong topology is stronger than the weak topology, the σ-strong topology is stronger than the strong topology and the σ-weak topology is stronger than the weak topology. In general σ-weak and strong topology are not comparable. Moreover, σ-strong and strong (resp. σ-weak and weak) operator topologies agree on bounded sets.

It can be proved that, if M is a von Neumann algebra, then there is a unique Banach space X such that M is the dual of X, and the σ-weak topology on M is the weak* topology on M as the dual of X.

Definizione 2.1.1 *A von Neumann algebra M acting on the Hilbert space H is a weak operator closed self-adjoint subalgebra of $B(H)$*

Since the norm topology is stronger than the weak operator topology, a von Neumann algebra is in particular a C*-algebra. It can be shown that a von Neumann algebra always contains a multiplicative identity.

The strong (resp. σ-strong, weak, σ-weak) topology on a von Neumann algebra $M \subset B(H)$ is the subspace topology on M induced by the strong (resp. σ-strong, weak, σ-weak) topology on $B(H)$. We tend to identify two von Neumann algebras when they are *-isomorphic, even though they act on different Hilbert spaces. It has to be noted though that the weak and strong operator topologies induced on M by these two actions are in general different. It can be proved that, instead, σ-strong and σ-weak topology do not depend on the concrete representation of the von Neumann algebra, i.e. they are *intrinsic*. Moreover, any *-isomorphism of von Neumann algebras is a homeomorphism with respect to the σ-weak and σ-strong topologies.

The foundation of the theory of von Neumann algebras is the Bicommutant Theorem, proved by von Neumann in the '30s. If S is a subset of $B(H)$, its **commutant** is

$$S' = \{x \in B(H) \,|\, \forall y \in S,\, xy = yx\}.$$

The double commutant S'' of S is the commutant of the commutant of S.

Teorema 2.1.2 *If S is a self-adjoint subalgebra of $B(H)$, then S is weakly closed (i.e., it is a von Neumann algebra) iff $S'' = S$*

It follows that, if S is a self-adjoint subset of $B(H)$, then S'' is the smallest von Neumann algebra containing S.

The von Neumann Bicommutant theorem gives an easy way to define von Neumann algebras. For example, consider a countable discrete group Γ and the left regular representation $\lambda : \Gamma \to \mathcal{U}\left(\ell^2(\Gamma)\right)$, defined by

$$\lambda_\sigma(\delta_\gamma) = \delta_{\sigma\gamma}.$$

Then, the double commutant of $\lambda[\Gamma]$ is a von Neumann algebra $L\Gamma$, called the group von Neumann algebra of Γ. If Γ acts on a probability space with measure preserving transformations, one can define a unitary representation $\sigma : \Gamma \to L^2(X)$ by

$$(\sigma_\gamma f)(x) = f\left(\gamma^{-1}x\right)$$

and also the unitary representation $\sigma \otimes \lambda : \Gamma \to L^2(X) \otimes \ell^2(\Gamma)$, where λ is again the left regular representation. Since $L^\infty(X)$ acts naturally on $L^2(X)$, and hence on $L^2(X) \otimes \ell^2(\Gamma)$, setting

$$t(f \otimes \delta_\gamma) = (tf) \otimes \delta_\gamma$$

for every $\gamma \in \Gamma$, $t \in L^\infty(X)$ and $f \in L^2(X)$, one regard $L^\infty(X)$ as a subset of $B\left(L^2(X) \otimes \ell^2(\Gamma)\right)$ and hence consider the double commutant $L^\infty(X) \rtimes \Gamma$ of the set

$$L^\infty(X) \cup (\sigma \otimes \lambda)[\Gamma],$$

which is called *cross product* von Neumann algebra.

Another classic result is the Kaplanski density theorem

Teorema 2.1.3 *If A is a self-adjoint subalgebra of $B(H)$, then the unit ball of A'' is the strong closure of the unit ball of A, and the unit ball of $(A'')_{sa} = \{x \in A'' \,|\, x \text{ is self-adjoint}\}$ is the strong closure of the unit ball of*

$$A_{sa} = \{x \in A \,|\, x \textit{ is self-adjoint}\}.$$

The fact that the unit ball of $B(H)$ is compact in the weak operator topology is proved in a way similar to the fact that the unit ball of H is weakly compact, applying the Tychonoff theorem on products of compact spaces.

As a direct consequence of the fact that a von Neumann algebra is strongly close, one can deduce that any upper bounded increasing net of self-adjoint elements in a von Neumann algebra M converges strongly to its.sup, which belongs to M. Moreover, every element x can be written uniquely in the form $u|x|$, where $|x| = (x^*x)^{\frac{1}{2}} \in M$ is the absolute value of x, i.e. the only positive element of $B(H)$ such that $\||x|\xi\| = \|x\xi\|$ for every $\xi \in H$, and $u \in M$ is the partial isometry such that u^*u is the projection onto $\overline{ran(|x|)} = \overline{ran(x^*)} = \ker(x)^{\perp}$ and uu^* is the projection onto $\overline{ran(x)}$.

2.2 Linear functionals on von Neumann algebras

The strongly and weakly continuous linear functionals on a von Neumann algebra admit a precise characterization.

Proposizione 2.2.1 *If $M \subset B(H)$ is a von Neumann algebra and ϕ a linear functional on M, TFAE*

1. *ϕ is weakly continuous*
2. *ϕ is strongly continuous*
3. *there are $\xi_1, ..., \xi_n, \eta_1, ..., \eta_n \in H$ such that, $\forall x \in M$,*

$$\phi(x) = \sum_{k=1}^{n} \langle x\xi_k, \eta_k \rangle$$

Proof.

$2 \Rightarrow 3$ By strong continuity, there are $\varepsilon > 0$ and $\xi_1, ..., \xi_n \in H$ such that, $\forall \xi \in M$, if $\|x\xi_i\| \leq \varepsilon$ for all $i \in \{1, 2, ..., n\}$, then $|\phi(x)| \leq 1$. Define $\mathfrak{H}$ the norm closure the subspace

$$\{(x\xi_1, ..., x\xi_n) \mid x \in M\}$$

in H^n. Define the bounded linear functional ψ on $\mathfrak{H}$ by

$$\psi(x\xi_1, ..., x\xi_n) = \phi(x)$$

Observe that ψ is well defined since, if $x\xi_i = 0$ for every $i \in \{1, 2, ..., n\}$, then $\phi(x) = 0$. By the Riesz-Fischer theorem, there are $\eta_1, ..., \eta_n \in H$ such that, for every $x \in M$,

$$\phi(x) = \psi(x\xi_1, ..., x\xi_n) = \sum_{i=1}^{n} \langle x\xi_i, \eta_i \rangle .$$

$3 \Rightarrow 1 \Rightarrow 2$ Obvious.

■

If, for $i \in \{1,2\}$, $M_i \subset B(H_i)$ is a von Neumann algebra, then the algebraic tensor product $M_1 \odot M_2$ acts on the Hilbertian tensor product $H_1 \otimes H_2$, by

$$(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta).$$

This gives an inclusion of $M_1 \odot M_2$ in $B(H_1 \otimes H_2)$. The strong closure of $M_1 \odot M_2$ in $B(H_1 \otimes H_2)$ is a von Neumann algebra, called the **tensor product** $M_1 \otimes M_2$ of M_1 and M_2.

For example, if $\mathbb{M}_n = B(\mathbb{C}^n)$ is the von Neumann algebra of $n \times n$ matrices with scalar coefficients and $M \subset B(H)$ is any von Neumann algebra, then $\mathbb{C}^n \otimes H \simeq H^n$ and $\mathbb{M}_n \otimes M \simeq \mathbb{M}_n(M) \subset B(H^n)$ is the von Neumann algebra of $n \times n$ matrices with coefficients in M.

The σ-strong and σ-weak topology on a von Neumann algebra $M \subset B(H)$ can be characterized in terms of tensor products. In fact, if 1 is the trivial von Neumann algebra in $B\left(l^2(\mathbb{N},\mathbb{C})\right)$, then

$$M \otimes 1 \subset B\left(H \otimes l^2(\mathbb{N},\mathbb{C})\right) = B\left(l^2(\mathbb{N},H)\right)$$

where, $\forall T \in M$, $\forall (\xi_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N},H)$,

$$(T \otimes 1)(\xi_n)_{n\in\mathbb{N}} = (T\xi_n)_{n\in\mathbb{N}},$$

is a von Neumann algebra isomorphic to M. The strong (resp. weak) weak operator topology on $M \otimes 1 \subset B\left(l^2(\mathbb{N},H)\right)$ is exactly (modulo the isomorphism $T \to T \otimes 1$) the σ-strong (res. σ-weak) operator topology on M.

The σ-weak and σ-strong functionals admit themselves a characterization.

Proposizione 2.2.2 *If ϕ is a linear functional on a von Neumann algebra M, TFAE*

1. $\exists (\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \in l^2(H)$ *such that,* $\forall x \in M$,

$$\phi(x) = \sum_{n\in\mathbb{N}} \langle x\xi_n, \eta_n \rangle$$

2. *ϕ is σ-weakly continuous*

3. *ϕ is σ-strongly continuous*

4. *ϕ is weakly continuous on the unit ball M_1 of M*

5. *ϕ is strongly continuous on the unit ball M_1 of M*

Proof.

$1 \Rightarrow 2$ Suppose $(x_i)_{i\in I}$ converges σ-weakly to 0. Thus $\left((x_i\xi_n)_{n\in\mathbb{N}}\right)_{i\in I}$ converges weakly to 0 in $\ell^2(\mathbb{N},H)$, and hence $(\phi(x_i))_{i\in I} = \left(\sum_{n\in\mathbb{N}} \langle x_i\xi_n, \eta_n \rangle\right)_{i\in I}$ converges to 0.

$2 \Rightarrow 3$ It follows from the fact that the σ-strong topology is stronger than the σ-weak topology.

$2 \Rightarrow 4$ It follows from the fact that σ-weak and weak operator topologies agree on bounded sets

$3 \Rightarrow 5$ It follows from the fact that σ-strong and strong operator topologies agree on bounded sets

$4 \Rightarrow 5$ It follows that the strong operator topology is stronger than the weak operator topology

$5 \Rightarrow 2$ The σ-weak topology is a weak* topology on M. Moreover, σ-weak and weak topology agree on bounded sets. The conclusion follows from the Krein-Smulian theorem.

$3 \Rightarrow 1$ By σ-weak continuity, there are $\xi^i = \left(\xi^i_n\right)_{n\in\mathbb{N}}$ for $i \in \{1, 2, ..., N\}$ and $\varepsilon > 0$ such that $\sum_n \left\|x\xi^i_n\right\|^2 < \varepsilon$ for every $i \in \{1, 2, ..., N\}$ implies $|\phi(x)| \leq 1$. Define $\mathfrak{H}$ the norm closure of

$$\left\{\left(x\xi^1, ..., x\xi^N\right) \mid x \in M\right\}$$

in $\ell^2(H, \mathbb{N} \times \{1, 2, ..., N\})$ and let ψ be the bounded linear functional on $\mathfrak{H}$ defined by

$$\psi\left(x\xi^1, ..., x\xi^N\right) = \phi(x).$$

By the Riesz-Fisher theorem, there is $\left(\eta^1, ..., \eta^n\right) \in \ell^2(H, \mathbb{N} \times \{1, 2, ..., N\})$ such that

$$\begin{aligned}
\phi(x) &= \psi\left(x\xi^1, ..., x\xi^N\right) \\
&= \left\langle\left(x\xi^1, ..., x\xi^N\right), \left(\eta^1, ..., \eta^N\right)\right\rangle \\
&= \sum_{k=1}^{N}\sum_{j\in\mathbb{N}}\left\langle x\xi^k_j, \eta^k_j\right\rangle.
\end{aligned}$$

The result follows observing that $\ell^2(H, \mathbb{N} \times \{1, 2, ..., N\}) \simeq \ell^2(H, \mathbb{N})$.

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Corollario 2.2.3 *If π is a *-representation of M on H, TFAE*

1. *π is **normal**, i.e. ultraweak-ultraweak continuous*
2. *the restriction of π to the unit ball is weak-weak continuous*
3. *the restriction of π to the unit ball is strong-weak continuous*
4. *π is ultrastrong-ultraweak continuous*

Proof.

$1 \Rightarrow 2$ Obvious

$2 \Rightarrow 3$ Obvious

$3 \Rightarrow 4$ Since π is a contraction and weak and ultraweak topologies coincide on bounded sets, $\pi_{|M_1}$ is weak-weak continuous.

$4 \Rightarrow 1$ Consider the representation Π of M on $\ell^2(H)$ defined by

$$\Pi(x)(\xi_n)_{n\in\mathbb{N}} = (\pi(x)\xi_n)_{n\in\mathbb{N}}.$$

Since π is a contraction and σ-strong and strong topology coincide on bounded sets, $\pi_{|M_1}$ is strong-ultraweak continuous. This implies that $\Pi_{|M_1}$ is strong-weak continuous. If $\xi, \eta \in \ell^2(H)$ and $\psi_{\xi,\eta}(T) = \langle T\xi, \eta\rangle$ for $T \in B(\ell^2(H))$, then $\psi_{\xi,\eta} \circ \Pi_{|M_1}$ is strongly continuous. Since $\psi_{\xi,\eta} \circ \Pi$ is a linear functional on M, this implies that $\psi_{\xi,\eta} \circ \Pi$ is ultraweakly continuous. Since this is true for every $\xi, \eta \in \ell^2(H)$, Π is ultraweak-weak continuous and hence π is ultraweak-ultraweak continuous.

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From the fact that (σ-)strong and (σ-)weak operator topologies have the same continuous functionals, it follows that a convex subset of $B(H)$ is (σ-)strongly closed iff it is (σ-)weakly closed.

The follow characterization of von Neumann algebras follows directly from this fact, the von Neumann bicommutant theorem and the Kaplanski density theorem.

Proposizione 2.2.4 *If A is a self-adjoint subalgebra of $B(H)$, TFAE*

1. *$A'' = A$*
2. *A is a weakly closed, i.e. it is a von Neumann algebra*
3. *A is strongly closed*
4. *A is σ-strongly closed*
5. *A is σ-weakly closed*
6. *the unit ball of A is weakly closed*
7. *the unit ball of A is strongly closed*

Proof.

$1 \Leftrightarrow 2$ It is the Bicommutant Theorem

$2 \Leftrightarrow 3$ It follows from the fact that a convex set is (σ-)strongly closed iff it is (σ-)weakly closed

$4 \Leftrightarrow 5$ Idem

$6 \Leftrightarrow 7$ Idem

$2 \Leftrightarrow 6$ It follows from the Krein-Smulian theorem and the fact that σ-weak and weak topology coincide on bounded sets

$3 \Rightarrow 4$ It follows from the fact that the σ-strong topology is stronger than the strong topology

$5 \Rightarrow 6$ It follows from the fact that the unit ball of $B(H)$ is strongly closed and σ-strong and strong topology coincide on bounded sets

■

Proposizione 2.2.5 *If ϕ is a state on a von Neumann algebra M, TFAE*

1. *ϕ is σ-weakly continuous*
2. *for every bounded decreasing net $(y_i)_{i\in I}$ in M_+ such that* $\inf_{i\in I} y_i = 0$, $\lim_{i\in I} \phi(y_i) = \inf_{i\in I} \phi(y_i) = 0$
3. *ϕ is **normal**, i.e. for every bounded increasing net $(x_i)_{i\in I}$ of self-adjoint elements of M,* $\lim_{i\in I} \phi(x_i)$ *exists and it is equal to* $\phi(\sup_i x_i)$

Proof.

$1 \Rightarrow 2$ The net $(y_i)_{i\in I}$ converges strongly to 0, and ϕ is strongly continuous on bounded sets.

$2 \Rightarrow 3$ Consider $x = \sup_i x_i$ and $y_i = x - x_i$ for every $i \in I$.

$3 \Rightarrow 2$ Consider $x_i = -y_i$ and observe that $\sup_i x_i = -\inf_i y_i = 0$

$2 \Rightarrow 1$ It is enough to prove that, if $(x_i)_{i\in I}$ is a net in M_1 strongly converging to 0, then $\lim_{i\in I} \phi(x_i) = 0$. If $(x_i)_{i\in I}$ is such a net, $\left(\frac{(x_i+x_i^*)}{2}\right)_{i\in I}$ and $\left(i\frac{x_i-x_i^*}{2}\right)_{i\in I}$ are nets of self-adjoint elements in M_1 strongly converging to 0. Therefore, without loss of generality, I can assume $x_i = x_i^*$ for every $i \in I$. In this case, $\left(\frac{x_i+|x_i|}{2}\right)_{i\in I}$ and $\left(\frac{|x_i|-x_i}{2}\right)_{i\in I}$ are nets of positive elements in M_1 strongly converging to 0. It is therefore enough considering the case $x_i \in M_+$ for every $i \in I$. In this case, suppose $y_i = \sup_{j\geq i} x_j$ and observe that $(y_i)_{i\in I}$ is a bounded decreasing net in M_+ such that $\inf_{i\in I} y_i = 0$. Thus, $\lim_i \phi(y_i) = 0$. Since $\phi(y_i) \geq \phi(x_i) \geq 0$, it follows that $\lim_{i\in I} \phi(x_i) = 0$.

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2.3 Projections and classification

A projection p of a von Neumann algebra M is a self-adjoint idempotent element of M. Projections have a central role in the study of von Neumann algebra.

Proposizione 2.3.1 *If M is a von Neumann algebra, then M is the norm closure of the set of its projections.*

If M is a von Neumann algebra and $p, q \in M$ are projections, we write

- $p \leq q$ if $pq = qp = p$
- $p \sim q$ if there is $u \in M$ such that $u^*u = p$ and $uu^* = q$
- $p \precsim q$ if $p \sim p' \leq q$.

It can be proved that $\precsim$ is a preorder whose induced equivalence relation is $\sim$.

A projection $p \in M$ is called

- **finite** if $p \sim q \leq p$ implies $q = p$
- **infinite** if it is not finite or, equivalently, there is a countably infinite orthogonal family of pairwise equivalent nonzero subprojections of p
- **properly infinite** if $p \sim p_1 + p_2$, $p_1 \sim p \sim p_2$ and $p_1 \perp p_2$ or, equivalently, p is the sum of a countably infinite orthogonal family of subprojections of p isomorphic to p
- **purely infinite** if it does not contain any nonzero finite projection
- **semifinite** if does not contain any purely infinite projection or, equivalently, every nonzero subprojection of p contains a nonzero finite projection
- **abelian** if pMp is abelian
- **minimal** if it is minimal with respect to the $\precsim$ preorder
- **continuous** if it does not contain any nonzero abelian projection
- **discrete** if does not contain any nonzero continuous subprojection or, equivalently, every nonzero subprojection of p contains a nonzero abelian projection

It is easily seen that these properties are preserved under equivalence. Moreover, any abelian projection and any subprojection of a finite projection is finite.

A von Neumann algebra M is called finite (infinite, purely infinite, etc...) if the identity 1_M of M is finite (infinite, purely infinite, etc...).

A von Neumann algebra is called of

- **type I** if it is **discrete**

- **type II$_1$** if is **continuous and finite**
- **type II$_\infty$** if it is **semifinite, continuous and infinite**
- **type III** if it is **purely infinite**

Maximality arguments show that, if M is a von Neumann algebra, then M can be written as a direct sum of

- a finite and a properly infinite von Neumann algebra
- a semifinite and a purely infinite von Neumann algebra
- a discrete and a continuous von Neumann algebra

From this result , it is not difficult to deduce the classification theorem for von Neumann algebras: any von Neumann algebra can be written as a direct sum

$$M = M_I \oplus M_{II_1} \oplus M_{II_\infty} \oplus M_{III}$$

where $M_I, M_{II_1}, M_{II_\infty}, M_{III}$ are von Neumann algebras of type I,II_1,II_∞ and III respectively. Moreover, M_I can be written as a finite direct sum of an infinite discrete (i.e. of type I_∞) von Neumann algebra M_{I_∞} isomorphic to $B(H) \otimes Z$ and von Neumann algebras M_{I_n} isomorphic to $\mathbb{M}_n \otimes Z_n$ (i.e. of type I_n) for $n \in \mathbb{N}$, where H is a Hilbert space, $\mathbb{M}_n$ is the algebra of $n \times n$ matrixes over $\mathbb{C}$ and Z, Z_n are commutative von Neumann algebras.

It can be proved that a von Neumann algebra M is finite if and only if it has a tracial state τ. Moreover, if M is a finite factor, such a tracial state is unique and it turns out to be normal, faithful and such that, for every $x \in M$, $\tau(x)1$ is the only element of $\mathbb{C}1 \subset M$ which belongs to the norm closed convex hull of $\{uxu^* \mid u \in M \text{ unitary}\}$ (Diximier property).

For a projection p is equivalent being finite and the fact that, whenever q, r are equivalent subprojections of p, also $p-q$ and $p-r$ are equivalent subprojections of p. This implies that, in a finite von Neumann algebra, a parial isometry u in M such that $u^*u = 1$ is a unitary. Moreover, any two equivalent projections are conjugate by a unitary and, as a consequence, every $x \in M$ has a polar decomposition of the form $u|x|$ where u is a unitary.

2.4 Factors and dimension function

Definizione 2.4.1 *The center $Z(M)$ of a von Neumann algebra M is $M \cap M'$. A von Neumann algebra with trivial center is called a factor.*

Every von Neumann algebra can be decomposed into an integral of factors. Therefore, in order to study von Neumann algebras, it is no loss of generality restricting to factors. By the classification theorem, a finite factor is either of type II_1 or of type I_n for some $n \in \mathbb{N}$.

If M is a factor, the pre-order $\precsim$ is total, and a projection is abelian if and only if it is finite. It follows that, in a factor, any two minimal projections are equivalent. Moreover, a type II_1 factor is a finite factor that does not contain any minimal projection. Instead, a finite type I factor M is a factor such that any projection contains a minimal projection. Since any two minimal projections are conjugate and orthogonal, M has only finitely many minimal projections. If n is the number of minimal projections of M, then M is of type I_n, i.e. isomorphic to the algebra $\mathbb{M}_n$ of $n \times n$ matrices over $\mathbb{C}$.

If M is a finite factor, there is a unique function d, called **dimension function**, from the set of projections of M to $[0,1]$ such that $d(1) = 1$, $d(p+q) = d(p) + d(q)$ if $p \perp q$ and $d(p) \leq d(q)$ iff $p \precsim q$. Moreover, if τ is the trace of M, then $\tau(p) = d(p)$ for every projection p. In fact, if $M = \mathbb{M}_n$ is of type I_n, then it is clear that

$$d(p) = \dim rank(p)$$

is the unique function having those properties. Moreover, the trace τ of $\mathbb{M}_n$ is the usual matrix trace, and it is cleat that in this case $d(p) = \tau(p)$ for every projection p. Suppose now that M is a II_1 factor. Since $\precsim$ is total in M and M has no minimal projections, it follows that any nonzero projection of M has nonzero equivalent orthogonal subprojections. Define $p_1 = 1$ and $p_0 = 0$. Suppose $\{p_i, q_i\}_{i \in I}$ is a maximal orthogonal family of projections of M such that $p_i \sim q_i$ for every $i \in I$. Define

$$p_{\frac{1}{2}} = \sum_{i \in I} p_i$$

and

$$q_{\frac{1}{2}} = \sum_{i \in I} q_i.$$

Then, $p_{\frac{1}{2}} \sim q_{\frac{1}{2}}$, $p_{\frac{1}{2}}, q_{\frac{1}{2}}$ are orthogonal and $p_{\frac{1}{2}} + q_{\frac{1}{2}} = 1$. Analogously, find $p_{\frac{1}{4}} \leq p_{\frac{1}{2}}$ such that $p_{\frac{1}{4}} \sim \left(p_{\frac{1}{2}} - p_{\frac{1}{4}}\right)$ and define, if u is a partial isometry such that $u^*u = p_{\frac{1}{2}}$ and $uu^* = p - p_{\frac{1}{2}}$,

$$p_{\frac{3}{4}} = p_{\frac{1}{2}} + up_{\frac{1}{4}}u^*.$$

Proceeding inductively, it is possible to define, for every diadic rational α, a projection p_α in M such that the function $\alpha \to p_\alpha$ is monotone and, if $\alpha \leq \beta$, then $p_\beta - p_\alpha \sim p_{\beta - \alpha}$. Define now, for $\alpha \in [0,1]$,

$$p_\alpha = \sup \{p_\beta \mid \beta \text{ diadic rational } \leq \alpha\}.$$

It is clear that, if $\alpha < \beta$, then $p_\alpha < p_\beta$. Observe that, if p is a nonzero projection, then $p_\alpha \precsim p$ for sufficiently small α. If not, then $p \precsim p_\alpha$ for every α and hence, $p_\alpha \precsim p_{2^{-n+1} - 2^n} = p_{2^{-n+1}} - p_{2^n}$ for every $n \in \mathbb{N}$. This implies that there is an infinite orthogonal family of nonzero equivalent projections of M, contradicting finiteness of M. It follows that

$$\inf \{p_\beta \mid \beta \in [0,1]\} = 0.$$

If $\alpha \in (0,1)$ and

$$p'_\alpha = \inf \{ p_\beta \,|\, \beta \in (\alpha, 1] \}$$

then

$$p'_\alpha - p_\alpha \leq p_\beta - p_\alpha \sim p_{\beta-\alpha}$$

for every $\beta \in (\alpha, 1]$, and hence $p'_\alpha = p_\alpha$. Now, if p is a projection in M and

$$\alpha = \inf \{ \beta \in [0,1] \,|\, p \precsim p_\beta \} = \sup \{ \gamma \in [0,1] \,|\, p_\gamma \precsim p \}$$

then $p \sim p_\alpha$. In fact, by comparability, $p \precsim p_\alpha$ or $p_\alpha \precsim p$. Suppose $p \precsim p_\alpha$. Thus, there is $q \leq p_\alpha$ such that $q \sim p$. Now, if $\beta < \alpha$ then $p_\beta \leq p \sim q$ implies

$$p_\alpha - q \precsim p_\alpha - p_\beta \sim p_{\beta-\alpha}$$

and, since this is true for every $\beta < \alpha$, $p_\alpha = q \sim p$. If $p_\alpha \precsim p$ then $1 - p \precsim 1 - p_\alpha \sim p_{1-\alpha}$, where

$$1 - \alpha = \inf \{ \beta \in [0,1] \,|\, 1 - p \precsim p_\beta \} = \sup \{ \gamma \in [0,1] \,|\, p_\gamma \precsim 1 - p \}$$

and hence, by the previous case, $1 - p \sim p_{1-\alpha} \sim 1 - p_\alpha$ and $p \sim p_\alpha$. It is clear that a dimension d has to be such that $d(p_\alpha) = \alpha$ for every $\alpha \in [0,1]$ and such a function is in fact a dimension function. If now τ is the unique (faithful normal) tracial state on M, it is easily proved by induction that $\tau(d_\alpha) = \alpha$ for every diadic rational α and hence, by normality, $\tau(d_\alpha) = \alpha$ for every $\alpha \in [0,1]$.

Lemma 2.4.2 *If M is a II_1 factor and $n \in \omega$, then there is an injective *-homomorphism $\Phi : \mathbb{M}_{2^n} \to M$ such that $\tau_M \circ \Phi = \tau_{\mathbb{M}_{2^n}}$*

Proof. By induction on $n \in \omega$. If $n = 0$ then $\mathbb{M}_{2^n}$ is the trivial algebra. Suppose the thesis is true for n and identify $\mathbb{M}_{2^{n+1}}$ with $\mathbb{M}_2 \otimes \mathbb{M}_{2^n}$. Suppose $p \in M$ is a projection such that $\tau(p) = \frac{1}{2}$. By inductive hypothesis, there is an injective *-homomorphism $\Phi : \mathbb{M}_{2^n} \to pMp$ such that $\tau(\Phi(x)) = \frac{\tau_{\mathbb{M}_{2^n}}(x)}{2}$ for every $x \in \mathbb{M}_{2^n}$. Suppose $u \in M$ is a partial isometry whose range projection is p and source projection in $1-p$. Thus, $u^*u = u^*(1-p)u = p$ and $uu^* = upu^* = 1-p$,

$$u(pMp)u^* = (1-p)M(1-p)$$

$$u(pMp)u = (1-p)Mp$$

and

$$u^*(pMp)u^* = pM(1-p).$$

Define now the function $\Psi : \mathbb{M}_2 \otimes \mathbb{M}_{2^n}$ by

$$\Psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes x\right) = a\Psi(x) + bu\Psi(x)u + cu^*\Psi(x)u^* + du\Psi(x)u^*$$

It is easily seen that Ψ is a *-homomorphism. Moreover,

$$\begin{aligned}\tau_M\left(\Psi\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\otimes x\right)\right) &= \tau_M\left(a\Psi(x)+bu\Psi(x)u+cu^*\Psi(x)u^*+du\Psi(x)u^*\right)\\ &= a\tau_M(\Psi(x))+d\tau_M(u\Psi(x)u^*)\\ &= \frac{1}{2}(a+d)\tau_{\mathbb{M}_{2^n}}(x) = \\ &= \tau_{\mathbb{M}_2}\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)\tau_{\mathbb{M}_{2^n}}(x)\\ &= \tau_{\mathbb{M}_{2^{n+1}}}(x)\end{aligned}$$

■

2.5 Tracial states

A **tracial** von Neumann algebra is a (finite) von Neumann algebra M endowed with a faithful normal tracial state τ. A tracial von Neumann algebra can be endowed with a scalar product $\langle x, y\rangle_\tau = \tau(y^*x)$. The completion of M with respect to this scalar product is denoted by $L^2(M,\tau)$. Define $x \to \widehat{x}$ the natural embedding of M into $L^2(M,\tau)$. Define, for $x \in M$, $\pi(x) \in B\left(L^2(M,\tau)\right)$ such that

$$\pi(x)\widehat{y} = \widehat{xy}$$

for every $y \in M$. Observe that, $\forall x, y, z \in M$,

$$\begin{aligned}\|\widehat{xy}\|_2^2 &= \tau(y^*x^*xy)\\ &\leq \tau\left(y^*\|x^*x\|_\infty y\right)\\ &= \|x^*x\|_\infty\|y\|_2^2\\ &= \|x\|_\infty^2\|y\|_2^2\end{aligned}$$

and hence $\pi(x) \in B\left(L^2(M,\tau)\right)$ is well defined. Observe that $\widehat{1}$ is a separating vector, i.e. $\pi(x)\widehat{1} = 0$ implies $x = 0$. In particular, π is injective. Moreover,

$$\begin{aligned}\langle\pi(x^*)\widehat{y},\widehat{z}\rangle &= \left\langle\widehat{x^*y},\widehat{z}\right\rangle\\ &= \tau(z^*x^*y)\\ &= \tau\left((xz)^*y\right)\\ &= \langle\widehat{y},\widehat{xz}\rangle\\ &= \langle\widehat{y},\pi(x)\widehat{z}\rangle\\ &= \left\langle\pi(x)^*\widehat{y},\widehat{z}\right\rangle\end{aligned}$$

and hence π is a $*$-homomorphism. Being injective, its image is a C*-algebra and π is an isometry with respect to $\|\cdot\|_\infty$. I now claim that π is normal. By a previous corollary, it is enough to prove that $\pi_{|M_1}$ is weak-weak continuous.

Suppose thus that $(z_i)_{i\in I}$ is a net in M_1 weakly converging to 0. I have to prove that $(\pi(z_i))_{i\in I}$ converges weakly to 0. Since $(\pi(z_i))_{i\in I}$ is bounded, it is enough to prove that $(\langle \pi(z_i)\widehat{x},\widehat{y}\rangle)_{i\in I}$ converges to 0 for all $x,y\in M$. We have

$$\langle \pi(z_i)\widehat{x},\widehat{y}\rangle = \tau(y^* z_i x)$$

where $(y^* z_i x)_{i\in I}$ converges weakly to 0, and hence by normality of τ,

$$0=\lim_i \tau(y^* z_i x)=\lim_i \langle \pi(z_i)\widehat{x},\widehat{y}\rangle$$

This implies that $\pi(M)$ is itself a von Neumann algebra. In fact, by (σ-)weak compactness of M_1 and σ-weak continuity of π, $\pi(M_1)=(\pi(M))_1$ is (σ-)weak compact and hence $\pi(M)$ is a von Neumann algebra. This shows that M can be regarded as a von Neumann algebra acting on $L^2(M,\tau)$.

From this fact, it can deduced, for example, that the topology induced by the $\|\cdot\|_2$ on M_1 coincides with the (σ-)strong topology. In fact, regard M as a subset of $B(L^2(M,\tau))$. If $(z_i)_{i\in I}$ is a net in M_1 converging to 0 in $\|\cdot\|_2$, I claim that $(z_i)_{i\in I}$ converges strongly to 0 (wrt the action of M on $L^2(M,\tau)$). Since $(z_i)_{i\in i}$ is a bounded net and $\{\widehat{y}\,|\,y\in M\}$ is a dense subspace of $L^2(M,\tau)$, it is enough to prove that $\lim_{i\in I}\|z_i\widehat{y}\|_2=0$ for every $y\in M$. We have

$$\begin{aligned}\lim_i \|z_i\widehat{y}\|_2 &= \lim_i \|z_i y\|_2\\ &\leq \|y\|_\infty \lim_i \|z_i\|_2 = 0\end{aligned}$$

Conversely, if $(z_i)_{i\in I}$ converges to 0 strongly (wrt $L^2(M,\tau)$), then

$$0=\lim_{i\in I}\left\|z_i\widehat{1}\right\|_2=\lim_{i\in I}\|z_i\|_2=0.$$

It follows that the unit ball M_1 of M with the metric induced by the $\|\cdot\|_2$ is a complete metric space. In fact, suppose $(z_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_2$-Cauchy sequence in M_1. If $x\in M$, then

$$(\pi(z_n)\widehat{x})_{n\in\mathbb{N}}=(\widehat{z_n x})_{n\in\mathbb{N}}$$

is a Cauchy sequence in $L^2(M,\tau)$, and hence it converges to $T\widehat{x}\in L^2(M,\tau)$ such that $\|T\widehat{x}\|_2\leq\|\widehat{x}\|_2$. This defines a bounded linear operator T on $L^2(M,\tau)$ of norm ≤ 1, such that $(\pi(z_n))_{n\in\mathbb{N}}$ converges strongly to it. Since $\pi(M)$ is a von Neumann algebra, it is strongly closed and hence $T\in\pi(M)$ and $T=\pi(z)$ for some $z\in M_1$. We have

$$0=\lim_n\left\|\pi(z_n)\widehat{1}-\pi(z)\widehat{1}\right\|=\lim_n\|z_n-z\|_2$$

and hence $(z_n)_{n\in\mathbb{N}}$ converges to z in $\|\cdot\|_2$. Since this is true for every Cauchy sequence in $\|\cdot\|_2$, it follows that M_1 is complete wrt to the distance induced by $\|\cdot\|_2$.

Reasoning as above, it can be prove that, if M is a C*-algebra endowed with a faithful trace τ such that the norm unit ball of M is $\|\cdot\|_2$-complete, then M is a von Neumann algebra and τ is normal. In fact, if $\pi : M \to B\left(L^2(M,\tau)\right)$ is the GNS representation, as before the 2-topology on $\pi(M)_1 = \pi(M_1)$ coincide with the strong topology. Since $\pi(M)_1$ is a complete metric space wrt to the 2-norm, it has to be strongly closed. Moreover, if $(z_i)_{i\in I}$ is a net in M_1 strongly converging to 0 then $\left(z_i^{\frac{1}{2}}\right)_{i\in I}$ is a net in M_1 strongly converging to 0. Thus,

$$0 = \lim_{i\in I} \left\| z_i^{\frac{1}{2}} \right\|_2^2 = \lim_{i\in I} \tau(z_i)$$

and τ is normal. As a consequence, the GNS representation π is normal too.

Chapter 3

Ultraproducts

If I is a set, $(X_n, d_n)_{n\in I}$ is an I-sequence of uniformly bounded metric spaces and $\mathcal{U}$ an ultrafilter over I, define on $\prod_{n\in I} X_n$ the bounded pseudometric

$$\widetilde{d}^{\mathcal{U}}(x,y) = \mathcal{U} - \lim_{n\in I} d_n(x_n, y_n).$$

The bounded metric $\prod_n^{\mathcal{U}} X_n$space obtained from $\prod_n X_n$ and the pseudometric $\widetilde{d}^{\mathcal{U}}$, namely the quotient of $\prod_n X_n$ with respect to the equivalence relation

$$\mathbf{x} \sim_{\mathcal{U}} \mathbf{y}$$

iff

$$d^{\mathcal{U}}(\mathbf{x},\mathbf{y}) = 0$$

endowed with the metric,

$$d^{\mathcal{U}}([\mathbf{x}]_{\mathcal{U}}, [\mathbf{y}]_{\mathcal{U}}) = \widetilde{d}^{\mathcal{U}}(\mathbf{x},\mathbf{y}),$$

is called the **metric ultraproduct** of the sequence $(X_n, d_n)_{n\in I}$ of metric spaces with respect to the ultrafilter $\mathcal{U}$ over $\mathbb{N}$. It is worth noting that, if each one (or just $\mathcal{U}$-a.a.) of the (X_n, d_n) is complete, then $\prod_n^{\mathcal{U}} X_n$ is complete. In fact, suppose $(\mathbf{x}^n)_{n\in I}$ is a Cauchy sequence in $\prod_n^{\mathcal{U}} X_n$ where, without loss of generality, $\forall n \in \mathbb{N}$, $d^{\mathcal{U}}(\mathbf{x}_n, \mathbf{x}_{n+1}) < 2^{-n}$. Define, $\forall n \in \mathbb{N}$, if $\mathbf{x}^n = \left[(x_k^n)_{k\in I}\right]_{\mathcal{U}}$,

$$A_n = \left\{k \in I \,\middle|\, \forall i \leq n,\, d\left(x_k^i, x_k^{i+1}\right) < 2^{-i}\right\}.$$

Observe that $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of elements of $\mathcal{U}$. Define now, for every $k \in I$, $y_k = x_k^n$ if $k \in A_n \backslash A_{n+1}$. If $k \in \bigcap_n A_n$, then

$$(x_k^n)_{n\in\mathbb{N}}$$

is a Cauchy sequence in (X_k, d_k), and hence it has a limit x_k. In this case, define $y_k = x_k$. Since $A_1 \in \mathcal{U}$, it is well defined the element $\mathbf{y} = \left[(y_k)_{k\in I}\right]$ of

$\prod_n^{\mathcal{U}} X_n$. I claim that $\mathbf{y}$ is the limit of the sequence $(\mathbf{x}^n)_{n\in\mathbb{N}}$. In fact, for every $n \in \mathbb{N}$ and $k \in A_n$,

$$d\left(y_k, x_k^n\right) \leq 2^{-n}$$

and hence

$$d^{\mathcal{U}}\left(\mathbf{y}, \mathbf{x}^n\right) \leq 2^{-n}.$$

Suppose now that $(H_n)_{n\in\mathbb{N}}$ is a sequence of Hilbert spaces and $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$. Define

$$\ell^\infty \left(H_n\right)_{n\in\mathbb{N}} = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} H_n \,\middle|\, \sup_n \|x_n\| < +\infty \right\}.$$

Then, $\ell^\infty \left(H_n\right)_{n\in\mathbb{N}}$ is a Banach space with norm

$$\left\|(x_n)_{n\in\mathbb{N}}\right\| = \sup_{n\in\mathbb{N}} \|x_n\|.$$

Define, for $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \ell^\infty \left(H_n\right)_{n\in\mathbb{N}}$,

$$\left\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \right\rangle = \mathcal{U} - \lim_n \langle x_n, y_n\rangle$$

This defines a sesquilinear form on $\ell^\infty \left(H_n\right)_{n\in\mathbb{N}}$ and a scalar product on the quotient $\prod_n^{\mathcal{U}} H_n$ of $\ell^\infty \left(H_n\right)_{n\in\mathbb{N}}$ with respect to the closed subspace

$$\left\{ (x_n)_{n\in\mathbb{N}} \,\middle|\, \mathcal{U} - \lim_n \|x_n\| = 0 \right\}.$$

I claim that the scalar product space obtained in this way is a Hilbert space. In fact, the unit ball of $\prod_n^{\mathcal{U}} H_n$ as a bounded metric space with respect to the distance induced by the norm, is a closed subset of the metric ultraproduct of the balls of radius 2 of the H_n's as uniformly bounded metric spaces with respect to the distances induced by the norms. Therefore, it is complete and, hence, $\prod_n^{\mathcal{U}} H_n$ is complete too. The Hilbert space $\prod_n^{\mathcal{U}} H_n$ constructed in this way is called the **ultrapower** of the sequence $(H_n)_{n\in\mathbb{N}}$ of Hilbert spaces with respect to the ultrafilter $\mathcal{U}$.

If $(A_n)_{n\in\mathbb{N}}$ is a sequence of C*-algebras and $\mathcal{U}$ a ultrafilter over $\mathbb{N}$, consider

$$\ell^\infty \left(A_n\right)_{n\in\mathbb{N}} = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_n M_n \,\middle|\, \sup_n \|x_n\| < +\infty \right\}$$

endowed with the norm

$$\left\|(x_n)_{n\in\mathbb{N}}\right\| = \sup_n \|x_n\|.$$

and pointwise operations. Then, $\ell^\infty \left(A_n\right)_{n\in\mathbb{N}}$ is a Banach algebra and

$$\mathcal{J}_{\mathcal{U}} = \left\{ (a_n)_{n\in\mathbb{N}} \in \ell^\infty \left(A_n\right)_{n\in\mathbb{N}} \,\middle|\, \mathcal{U} - \lim_{n\in\mathbb{N}} \|a_n\| = 0 \right\}$$

is a closed bilateral ideal of $\ell^{\infty}(A_n)_{n\in\mathbb{N}}$. The quotient Banach algebra $\ell^{\infty}(A_n)_{n\in\mathbb{N}}/\mathcal{J}_{\mathcal{U}}$ turns out to be a C*-algebra with involution

$$\left[(a_n)_{n\in\mathbb{N}}\right]^* = \left[(a_n^*)_{n\in\mathbb{N}}\right].$$

Suppose now that (M_n, τ_n) is a sequence of tracial von Neumann algebras and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. Then,

$$\ell^{\infty}(M_n)_{n\in\mathbb{N}} = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_n M_n \,\middle|\, \sup_n \|x_n\|_{\infty} < +\infty \right\}$$

is a Banach algebra with respect to the pointwise operations and the norm

$$\left\|(x_n)_{n\in\mathbb{N}}\right\| = \sup_n \|x_n\|_{\infty}.$$

Moreover,

$$\mathcal{I}_{\mathcal{U}} = \left\{ (x_n)_{n\in\mathbb{N}} \,\middle|\, \mathcal{U} - \lim_{n\in\mathbb{N}} \|x_n\|_2 = 0 \right\}$$

is a norm-closed two-sided ideal of $\ell^{\infty}(M_n)_{n\in\mathbb{N}}$. In fact, if $(x_n)_{n\in\mathbb{N}} \in \mathcal{I}_{\mathcal{U}}$ and $(y_n)_{n\in\mathbb{N}} \in \ell^{\infty}(M)$, then

$$\mathcal{U} - \lim_{n\in\mathbb{N}} \|x_n y_n\|_2 \leq \left(\sup_n \|y_n\|_{\infty}\right) \mathcal{U} - \lim_n \|x_n\|_2 = 0$$

and hence $(x_n y_n)_{n\in\mathbb{N}} \in \mathcal{I}_{\mathcal{U}}$ and, analogously, $(y_n x_n)_{n\in\mathbb{N}} \in \mathcal{I}_{\mathcal{U}}$. If $(y_n)_{n\in\mathbb{N}}$ belongs to the closure of $\mathcal{I}_{\mathcal{U}}$ and $\varepsilon > 0$, there is $(x_n)_{n\in\mathbb{N}} \in \mathcal{U}$ such that $\sup_n \|x_n - y_n\| < \frac{\varepsilon}{2}$. Thus,

$$\{n \in \mathbb{N} \mid \|y_n\|_2 < \varepsilon\} \supset \left\{ n \in \mathbb{N} \,\middle|\, \|x_n\|_2 < \frac{\varepsilon}{2} \right\} \in \mathcal{U}$$

and, since this is true for every $\varepsilon > 0$, $\mathcal{U} - \lim_n \|y_n\|_2 = 0$ and $(y_n)_{n\in\mathbb{N}} \in \mathcal{I}_{\mathcal{U}}$. Since $\mathcal{I}_{\mathcal{U}}$ is a norm-closed two-sided ideal, it is possible to consider the quotient Banach algebra $M^{\mathcal{U}} = \frac{\ell^{\infty}(M)}{\mathcal{I}_{\mathcal{U}}}$.

Define, for $\left[(x_n)_{n\in\mathbb{N}}\right]$,

$$\tau^{\mathcal{U}}\left(\left[(x_n)_{n\in\mathbb{N}}\right]\right) = \mathcal{U} - \lim_n \tau_n(x_n).$$

Observe that τ is a well defined linear functional. In fact, if $(x_n)_{n\in\mathbb{N}} \in \mathcal{I}_{\mathcal{U}}$ then

$$\begin{aligned}
\left|\mathcal{U} - \lim_n \tau(x_n)\right| &= \mathcal{U} - \lim_n |\tau_n(x_n)| \\
&= \mathcal{U} - \lim_n \left|\left\langle \widehat{x}_n, \widehat{1} \right\rangle\right| \\
&\leq \mathcal{U} - \lim_n \|x_n\|_2^2 = 0
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
\left|\tau^{\mathcal{U}}\left(\left[(x_n)_{n\in\mathbb{N}}\right]\right)\right| &= \left|\mathcal{U}-\lim_n \tau_n(x_n)\right| \\
&= \mathcal{U}-\lim_n |\tau_n(x_n)| \\
&\leq \mathcal{U}-\lim_n \|x_n\| \\
&\leq \sup_n \|x_n\|
\end{aligned}$$

and hence, also

$$\left|\tau^{\mathcal{U}}\left(\left[(x_n)_{n\in\mathbb{N}}\right]\right)\right| \leq \left\|\left[(x_n)_{n\in\mathbb{N}}\right]\right\|$$

and $\tau^{\mathcal{U}}$ is bounded of norm ≤ 1. Moreover, if $\left[(x_n)_{n\in\mathbb{N}}\right] \in M^{\mathcal{U}}$ is positive, then $x_n = y_n^2 + z_n$, with $\mathcal{U}-\lim_n \|z_n\|_2 = 0$ for $\mathcal{U}$-a.a. $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
\tau([(x_n)]) &= \mathcal{U}-\lim_n \tau^n(x_n) \\
&= \mathcal{U}-\lim_n \tau^n\left(y_n^2\right) + \mathcal{U}-\lim_n \tau^n(z_n) \\
&= \mathcal{U}-\lim_n \|y_n\|_2^2 \geq 0
\end{aligned}$$

If, moreover,

$$\mathcal{U}-\lim_n \|y_n\|_2^2 = \tau([(x_n)]) = 0$$

then $(y_n)_{n\in\mathbb{N}} \in \mathcal{I}_{\mathcal{U}}$ and $[(x_n)] = 0$. This shows that τ is positive and faithful. It is clear that $\tau^{\mathcal{U}}(1) = 1$ and $\tau^{\mathcal{U}}$ is tracial. Therefore, $\tau^{\mathcal{U}}$ is a faithful normal tracial state on $\prod_n^{\mathcal{U}} M_n$, and define an inner product

$$\langle x, y\rangle = \tau(y^* x)$$

on $\prod_n^{\mathcal{U}} M_n$, whose associated norm is

$$\|x\|_2^2 = \tau(x^* x).$$

Observe that the ∞-unit ball of $\prod_n^{\mathcal{U}} M_n$ endowed with the distance induced by the 2-norm coincides with the metric ultraproduct of the ∞-balls of radius 1 of the M_n's endowed with the metric induced by their 2-norms. Therefore, it is a complete metric space. It follows that $\prod_n^{\mathcal{U}} M_n$ is a von Neumann algebra and τ is normal on M.

I will now prove that $\prod_n^{\mathcal{U}} M_n$ admits a normal faithful representation on the closed subspace of $\prod_n^{\mathcal{U}} L^2(M_n, \tau_n)$ generated by

$$\left\{\left[(\widehat{x}_n)_{n\in\mathbb{N}}\right] \in \prod_n{}^{\mathcal{U}} L^2(M_n, \tau_n) \,\middle|\, (x_n)_{n\in\mathbb{N}} \in \ell^\infty(M_n)_{n\in\mathbb{N}}\right\}.$$

Define, for every $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \ell^\infty(M_n)_{n\in\mathbb{N}}$, if $\left[(\widehat{y}_n)_{n\in\mathbb{N}}\right]$ is the corresponding element in $\mathcal{H}$,

$$\widetilde{\rho}\left((x_n)_{n\in\mathbb{N}}\right)\left[(\widehat{y}_n)_{n\in\mathbb{N}}\right] = \left[(\widehat{x_n y_n})_{n\in\mathbb{N}}\right].$$

Observe that $\widetilde{\rho}\left((x_n)_{n\in\mathbb{N}}\right)\left[(\widehat{y}_n)_{n\in\mathbb{N}}\right]$ is well defined, since, if $\left[(\widehat{y}_n)_{n\in\mathbb{N}}\right]=0$ then $\mathcal{U}-\lim_n \|y_n\|_2=0$ and also

$$\mathcal{U}-\lim \|x_n y_n\|_2 \leq \sup_k \|x_k\|_\infty \mathcal{U}-\lim_n \|y_n\|_2 = 0$$

and hence $\left[(\widehat{x_n y_n})_{n\in\mathbb{N}}\right]=0$. Moreover, $\widetilde{\rho}\left((x_n)_{n\in\mathbb{N}}\right)$ is clearly linear and

$$\begin{aligned}
\left\|\widetilde{\rho}\left((x_n)_{n\in\mathbb{N}}\right)\left[(\widehat{y}_n)_{n\in\mathbb{N}}\right]\right\| &= \left\|\left[(\widehat{x_n y_n})_{n\in\mathbb{N}}\right]\right\| \\
&= \mathcal{U}-\lim_n \|x_n y_n\| \\
&\leq \sup_k \|x_k\|_\infty \mathcal{U}-\lim_n \|y_n\|_2 \\
&= \left\|(x_k)_{k\in\mathbb{N}}\right\| \left\|\left[(y_n)_{n\in\mathbb{N}}\right]\right\|
\end{aligned}$$

Thus, $\widetilde{\rho}\left((x_n)_{n\in\mathbb{N}}\right)$ can be extended to a bounded linear operator on $\mathcal{H}$ of norm $\leq \left\|(x_n)_{n\in\mathbb{N}}\right\|$. Observe also that $\widetilde{\rho}$ is an algebra homomorphism such that

$$\begin{aligned}
\left\langle \widetilde{\rho}\left((x_n)^*_{n\in\mathbb{N}}\right)\left[(\widehat{y}_n)_{n\in\mathbb{N}}\right], \left[(\widehat{z}_n)_{n\in\mathbb{N}}\right]\right\rangle &= \left\langle \widetilde{\rho}\left((x^*_n)_{n\in\mathbb{N}}\right)\left[(\widehat{y}_n)_{n\in\mathbb{N}}\right], \left[(\widehat{z}_n)_{n\in\mathbb{N}}\right]\right\rangle \\
&= \mathcal{U}-\lim_n \left\langle \widehat{x^*_n y_n}, \widehat{z}_n \right\rangle \\
&= \mathcal{U}-\lim_n \tau_n\left(z^*_n x^*_n y_n\right) \\
&= \mathcal{U}-\lim_n \tau_n\left((x_n z_n)^* y_n\right) \\
&= \mathcal{U}-\lim_n \langle \widehat{y_n}, \widehat{x_n z_n}\rangle \\
&= \left\langle \left[(\widehat{y}_n)_{n\in\mathbb{N}}\right], \widetilde{\rho}\left((x_n)_{n\in\mathbb{N}}\right)\left[(\widehat{z}_n)_{n\in\mathbb{N}}\right]\right\rangle \\
&= \left\langle \widetilde{\rho}\left((x_n)_{n\in\mathbb{N}}\right)^*\left[(\widehat{y}_n)_{n\in\mathbb{N}}\right], \left[(\widehat{z}_n)_{n\in\mathbb{N}}\right]\right\rangle
\end{aligned}$$

Thus, $\widetilde{\rho}$ is a *-representation of $\ell^\infty(M)$ on $\mathcal{H}$. Moreover, if $(x_n)_{n\in\mathbb{N}} \in \mathcal{I}_\mathcal{U}$ then, for every $(y_n)_{n\in\mathbb{N}} \in \ell^\infty(M_n)_{n\in\mathbb{N}}$,

$$\begin{aligned}
\left\|\widetilde{\rho}\left((x_n)_{n\in\mathbb{N}}\right)\left(\left[(y_n)_{n\in\mathbb{N}}\right]\right)\right\|^2 &= \mathcal{U}-\lim_n \|x_n y_n\|_2^2 \\
&= \mathcal{U}-\lim_n \tau_n\left(y^*_n x^*_n x_n y_n\right) \\
&= \mathcal{U}-\lim_n \tau_n\left(x_n y_n y^*_n x^*_n\right) \\
&\leq \mathcal{U}-\lim_n \|y_n\|_\infty^2 \|x_n\|_2^2 \\
&\leq \sup_k \|y_k\|_\infty \mathcal{U}-\lim_n \|x_n\|_2 = 0
\end{aligned}$$

and hence $\widetilde{\rho}\left((x_n)_{n\in\mathbb{N}}\right)=0$. Conversely, suppose that $\widetilde{\rho}\left((x_n)_{n\in\mathbb{N}}\right)=0$. Thus, if $y_n=1$ for every $n\in\mathbb{N}$,

$$\begin{aligned}
0 &= \left\|\widetilde{\rho}\left(\left[(x_n)_{n\in\mathbb{N}}\right]\right)\left[(y_n)_{n\in\mathbb{N}}\right]\right\| \\
&= \mathcal{U}-\lim_n \|x_n y_n\|_2 \\
&= \mathcal{U}-\lim_n \|x_n\|_2
\end{aligned}$$

and hence $(x_n)_{n\in\mathbb{N}} \in \mathcal{I}_{\mathcal{U}}$. This shows that $\widetilde{\rho}$ induces a faithful *-representation ρ of $\prod_n^{\mathcal{U}} M_n = \ell^\infty (M_n)_{n\in\mathbb{N}} / \mathcal{I}_{\mathcal{U}}$ on $\mathcal{H}$. In order to show that ρ is normal, i.e. ultraweak-ultraweak continuous, it is enough to show that the restriction of ρ to the ∞-unit ball is strong-weak continuous. Suppose thus that $(\mathbf{x}^i)_{i\in I}$ is a net in the ∞-unit ball of $\prod_n^{\mathcal{U}} M_n$ converging strongly (or, equivalently, in 2-norm) to 0. Thus,

$$\lim_{i\in I}\left(\mathcal{U}-\lim_n \left\|x_n^i\right\|_2^2\right)=0.$$

Without loss of generality, $x_n^i \in (M_n)_+$ for every $i \in I$ and $n \in \mathbb{N}$. Observe that, then $\left(\left[\left(\left(x_n^i\right)^{\frac{1}{2}}\right)_{n\in\mathbb{N}}\right]\right)_{i\in I}$ converges strongly (or in 2-norm) as well. If $(y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \in \ell^\infty (M_n)_{n\in\mathbb{N}}$, where $y_n, z_n \in (M_n)_+$, then

$$\begin{aligned}
\lim_{i\in I}\left\langle \rho\left(\mathbf{x}^i\right)\left[(\widehat{y}_n)_{n\in\mathbb{N}}\right], \left[(\widehat{z}_n)_{n\in\mathbb{N}}\right]\right\rangle &= \lim_{i\in I}\mathcal{U}-\lim_{n\in\mathbb{N}}\left\langle \widehat{x_n^i y_n}, \widehat{z}_n\right\rangle \\
&= \lim_{i\in I}\mathcal{U}-\lim_n \tau\left(z_n x_n^i y_n\right) \\
&\le \lim_{i\in I}\mathcal{U}-\lim_n \|z_n\|_\infty \|y_n\|_\infty \tau\left(x_n^i\right) \\
&\le \sup_{k\in\mathbb{N}}\|z_k\| \sup_{t\in\mathbb{N}}\|y_t\| \lim_{i\in I}\mathcal{U}-\lim_{n\in\mathbb{N}}\left\|\left(x_n^i\right)^{\frac{1}{2}}\right\|_2^2 = 0.
\end{aligned}$$

I will now prove that, if M_n is a factor for every $n \in \mathbb{N}$, then $\prod_n^{\mathcal{U}} M_n$ is a factor. Suppose by contradiction, that $\prod_n^{\mathcal{U}} M_n$ is not a factor, hence the center $\mathcal{Z}\left(\prod_n^{\mathcal{U}} M_n\right)$ of $\prod_n^{\mathcal{U}} M_n$ is nontrivial. Since a von Neumann algebra is the norm closure of its projections, there is a nontrivial projection $\mathbf{p} \in \mathcal{Z}\left(\prod_n^{\mathcal{U}} M_n\right)$. Since $\mathbf{p}$ is nontrivial, $\tau(p) = \alpha \in (0,1)$. Without loss of generality, in case replacing $\mathbf{p}$ with $1-\mathbf{p}$, I can assume $\alpha \in (0, \frac{1}{2}]$. Suppose $(p_n)_{n\in\mathbb{N}}$ is a representative of $\mathbf{p}$. Since

$$\tau(\mathbf{p}) = \mathcal{U}-\lim_{n\in\mathbb{N}} \tau_n(p_n) = \alpha,$$

then

$$\tau_n(p_n) \in \left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}\right)$$

for $\mathcal{U}$-a.a. $n \in \mathbb{N}$. Either $\tau_n(p_n) \le \frac{1}{2}$ for $\mathcal{U}$-a.a. $n \in \mathbb{N}$ or $\tau_n(p_n) \ge \frac{1}{2}$ for $\mathcal{U}$-a.a. $n \in \mathbb{N}$. In the first case, for $\mathcal{U}$-a.a. $n \in \mathbb{N}$, $\tau_n(p_n) \le \tau_n(1-p_n)$ and hence $p_n \precsim 1-p_n$ and hence there is a partial isometry u_n such that $u_n^* u_n = p_n$ and $u_n u_n^* = q_n \perp p_n$, then $\tau_n(u_n^* u_n - u_n u_n^*) \ge \frac{\alpha}{2}$. Defining $\mathbf{u} = \left[(u_n)_{n\in\mathbb{N}}\right] \in \prod_n^{\mathcal{U}} M_n$, one gets a partial isometry such that

$$\tau(\mathbf{u}^*\mathbf{u} - \mathbf{u}\mathbf{u}^*) = \mathcal{U}-\lim_n \tau_n(u_n^* u_n - u_n u_n^*) \ge \frac{\alpha}{2} > 0$$

and $\mathbf{u}\mathbf{u}^* \ne \mathbf{u}^*\mathbf{u}$. The case $\tau_n(p_n) \ge \frac{1}{2}$ for $\mathcal{U}$-a.a. can be reduced to the first one replacing p_n with $1 - p_n$. Since $\mathbf{p} = \mathbf{u}^*\mathbf{u}$ is central,

$$\begin{aligned} \mathbf{u}\mathbf{u}^* &= \mathbf{u}\mathbf{u}^*\mathbf{u}\mathbf{u}^* \\ &= \mathbf{u}\mathbf{p}\mathbf{u}^* \\ &= \mathbf{p}\mathbf{u}\mathbf{u}^* \end{aligned}$$

and

$$\mathbf{u}\mathbf{u}^* \leq \mathbf{u}^*\mathbf{u}$$

Moreover,

$$\mathbf{u}\mathbf{u}^* \sim \mathbf{u}^*\mathbf{u},$$

and hence, by finiteness $\mathbf{u}\mathbf{u}^* = \mathbf{u}^*\mathbf{u}$ and $\mathbf{u}\mathbf{u}^* - \mathbf{u}^*\mathbf{u} = 0$. Thus,

$$\begin{aligned} 0 &= \mathcal{U} - \lim_n \|u_n u_n^* - u_n^* u_n\|_2 = \\ &= \mathcal{U} - \lim_n \|p_n - q_n\|_2 \\ &= \mathcal{U} - \lim_n \tau\left(p_n + q_n\right) \\ &= 2 \cdot \mathcal{U} - \lim_n \tau\left(p_n\right) \geq \alpha > 0, \end{aligned}$$

which is a contradiction.

Part II

Sofic and hyperlinear groups

Chapter 4

Nonstandard methods

4.1 Superstructures

Let S be a set of **atoms**, i.e. elements that does not contain elements, and are taken as primitive (examples of S could be $\mathbb{N}$ or $\mathbb{R}$). Now, define inductively

$$S_0 = S$$

$$S_{n+1} = S_n \cup \wp(S_n)$$

and

$$\widehat{S} = \bigcup_{n\in\mathbb{N}} S_n$$

We say that $\widehat{S}$ is the **superstructure** of S. It turns out that the set $\widehat{S}$ contains virtually all mathematical objects that are needed in the practice when dealing with S, such as functions, topologies, measures etc.

In the following, we will refer to the elements of the superstructure which are not atoms as **sets**, and to elements of the superstructure which can be both atoms and sets as **entities**.

Proposizione 4.1.1 *Let S be a set of atoms and $\widehat{S}$ its superstructure. Then*

1. $\forall n \in \mathbb{N}$
$$S_n \in S_{n+1} \in \widehat{S}$$
and the S_n are sets of the superstructure

2. $\forall n \in \mathbb{N}$, *S_n is transitive, as well as $\widehat{S}$*

3. *if A is a set (of the superstructure) and $B \subseteq A$, then B is a set*

4. *if A is a set, $\wp(A)$ is a set*

5. *if $\mathcal{A}$ is a family of sets, then $\bigcap \mathcal{A}$ is a set and, if $\mathcal{A}$ is itself a set, then $\bigcup \mathcal{A}$ is a set*

6. *if* $A_1, ..., A_n$ *are sets, then* $A_1 \cup ... \cup A_n$ *and* $A_1 \times .... \times A_n$ *are sets*

7. *if* $x_1, ..., x_n \in \widehat{S}$*, then* $\{x_1, ..., x_n\}$ *is a set*

8. *all relations and functions on sets are sets*

Proposizione 4.1.2 *If* $n \in \mathbb{N}$*,* $A \in S_n$ *is a set and* $B \subseteq A$*, then* $B \in S_n$*.*

Proof. By induction on n. If $n = 0$ there's nothing to prove. If it is true for n and $A \in S_{n+1} = S_n \cup \wp(S_n)$ then, either $A \in S_n$, in which case $B \in S_n \subseteq S_{n+1}$ by induction hypothesis, or $A \in \wp(S_n)$. Hence also $B \in \acute{\wp}(S_n)$ and so $B \in S_{n+1}$. ■

4.2 Formulas

We assume the notion of formula of first order language as known. A formula is called a **sentence** if does not contain free variables. Here, as language, we consider the language $\mathcal{L}$ of set theory $\{\in\}$, with in addiction one simbol of constant $\underline{a}$ for each entity a of the superstructure $\widehat{S}$. A formula is said **bounded** if every quantifier is in the form

$$\forall x \in A$$

or

$$\exists x \in A$$

where A is either a constant or a variable.

Suppose that I is a map from $\widehat{S}$ to another superstructure $\widehat{T}$ and that α is an $\mathcal{L}$-formula. In the following, we will say that α is true if α is true with respect to the interpretation that assigns to each constant symbol $\underline{a}$ of $\mathcal{L}$ the corresponding element a of the superstructure $\widehat{S}$ and that interprets the symbol $\in$ as the usual set-theoretic relation of membership. Also, we will say that ${}^I\alpha$ is true if α is true with respect to the interpretation that assigns to each constant symbol $\underline{a}$ of $\mathcal{L}$ the corresponding element $I(a)$ of the superstructure $\widehat{T}$ and that interprets the symbol $\in$ as the usual relation of membership.

4.3 Elementary embeddings

Definizione 4.3.1 *A map* $* : \widehat{S} \to \widehat{T}$ *is called a* ***nonstandard map*** *if*

1. ${}^*S = T$

2. *for every infine set* $A \in \widehat{S}$*,* ${}^*A \neq \{{}^*a \mid a \in A\}$

3. $*$ *satisfies the* ***transfer principle****, namely a bounded formula* α *is true if and only of* ${}^*\alpha$ *is true*

Observe that, if A is a set of $\widehat{S}$ and we set

$${}^{\sigma}A = \{{}^{*}a \mid a \in A\},$$

then, by the transfer principle, ${}^{\sigma}A \subseteq {}^{*}A$. The second requirement for a nonstandard map ensures that this inclusion is proper.

In [?] it is shown that nonstandard maps actually exists, by means of the construction of *ultraproducts*.

4.4 Standard entities

Definizione 4.4.1 *An entity y of $\widehat{{}^{*}S}$ is called **internal standard** or an **hyperextension** if there is an entity x of $\widehat{S}$ such that ${}^{*}x = y$.*

Teorema 4.4.2 (Internal standard definition principle) *If ${}^{*}A$ is an internal standard set and α is a bounded formula with only free variable x and as constants $\underline{A}_1, ..., \underline{A}_n$, then*

$$\{x \in {}^{*}A \mid \alpha(x, {}^{*}A_1, ..., {}^{*}A_n)\} = {}^{*}\{x \in A \mid \alpha(x, A_1, ..., A_n)\}$$

is an internal standard set of $\widehat{{}^{*}S}$. Conversely, every internal standard set of $\widehat{{}^{*}S}$ can be written as above.

Proof. Let

$$B = \{x \in A \mid \alpha(x, A_1, ..., A_n)\}$$

and observe that

$$\forall x \in A\,(x \in B \longleftrightarrow \alpha(x, A_1, ..., A_n)).$$

Hence

$$\forall x \in {}^{*}A\,(x \in {}^{*}B \longleftrightarrow \alpha(x, {}^{*}A_1, ..., {}^{*}A_n))$$

$$B \subseteq A$$

and

$${}^{*}B \subseteq {}^{*}A.$$

Then

$${}^{*}A = \{x \in {}^{*}B \mid \alpha(x, {}^{*}A_1, ..., {}^{*}A_n)\}$$

because

$${}^{*}A = \{x \in {}^{*}A \mid x = x\}.$$

■

4.5 Superstructure monomorphisms

Definizione 4.5.1 *A map* $*: \widehat{S} \to \widehat{T}$ *is a* ***superstructure monomorphism*** *if it is one to one and*

1. *it preserves* $\in$ *and* $=$*:* $a \in A$ *iff* $^*a \in {}^*A$ *and* $a = b$ *iff* $^*a = {}^*b$
2. *it preserves finite sets:* $^*\{x_1, ..., x_n\} = \{^*x_1, ...,^* x_n\}$
3. *it preserves finite sequences:* $^*(x_1, ..., x_n) = (^*x_1, ...,^* x_n)$
4. *it preserves insiemistic operations:* $^*(A \cup B) = {}^*A \cup {}^*B$*,* $^*(A \cap B) = {}^*A \cap {}^*B$*,* $^*(A \backslash B) = {}^*A \backslash {}^*B$*,* $^*(A \times B) = {}^*A \times {}^*B$*,* $^*(\bigcup A) = \bigcup {}^*A$
5. *it preserves sections of relations: if* $\varphi \in A_1 \times ... \times A_n$ *is an* n*-ary relation and* $i \in \{1, 2, ..., n\}$*, then the set of* $x \in {}^*A_i$ *such that, for some* $a_1 \in {}^*A_1, ..., a_{i-1} \in {}^*A_{i-1}, a_{i+1} \in {}^*A_{i+1}, ...a_n \in {}^*A_n$*,*

   $$(a_1, .., a_{i-1}, x, a_{i+1}, ..., a_n) \in {}^*\varphi$$

   is the nonstandard extension of the set of $x \in A_i$ *such that, for some* $a_1 \in A_1, ..., a_{i-1} \in A_{i-1}, a_{i+1} \in A_{i+1}, ...a_n \in A_n$*,*

   $$(a_1, .., a_{i-1}, x, a_{i+1}, ..., a_n) \in \varphi$$

6. *it commutes with permutations of variables: if* $\varphi \in A_1 \times .... \times A_n$ *is an* n*-ary relation,* σ *a permutation of* $\{1, 2, ..., n\}$ *and* ψ *is the formula obtained by* φ *permuting the variables according to* σ*, namely* $(a_1, ..., a_n) \in \varphi$ *if and only* $(a_{\sigma(1)}, ..., a_{\sigma(n)}) \in \psi$*, then* $^*\psi$ *is the formula obtained by* φ *permuting the variables according to* σ

It can be easily seen by transfer that a map satisfying the transfer principle is s superstructure monomorphism. The nontrivial fact, which is proven for example in [**?**], is that also the converse is true.

We can apply the internal standard definition principle to relations, obtaining the following *internal standard definition principle for relations.*

Proposizione 4.5.2 *If* $B_1, ..., B_n \in \widehat{S}$ *and* α *is a bounded formula with constants* $\underline{C}_1, ..., \underline{C}_k$ *in with free variables* $x_1, ..., x_n$*, then*

$$\begin{aligned} &\{(x_1, ..., x_m) \in {}^*B_1 \times ... \times {}^*B_n \mid \alpha(x_1, ..., x_n,^* C_1, ...,^* C_k)\} = \\ = \;&{}^*\{(x_1, ..., x_n) \in B_1 \times ... \times B_n \mid \alpha(x_1, ..., x_n, C_1, ..., C_k)\} \end{aligned}$$

is an internal standard relation. Also, every n*-ary internal standard relation has this form.*

Proof. It follows from the standard definition principle and the fact that $*$ is a superstructure embedding. ■

The following theorem can be easily proved by means of the transfer principle.

Teorema 4.5.3 *If* $* : \widehat{S} \to \widehat{T}$ *is a nonstandard map,* $A, B, f \in \widehat{S}$ *and* f *is a function from* A *to* B *(and we write* $f : A \to B$*), then* ${}^*f$ *is a function from* ${}^*A$ *to* ${}^*B$*, and also*

1. f *is one-to-one iff* ${}^*f$ *is*

2. f *is onto iff* ${}^*f$ *is*

3. $dom\,({}^*f) = {}^*dom\,(f)$

4. $ran\,({}^*f) = {}^*ran\,(f)$

5. $\forall a \in A,\ {}^*\,(f\,(a)) = ({}^*f)\,({}^*a)$

6. *for all* $C \subseteq A,\ {}^*\left(f_{|C}\right) = ({}^*f)_{|C}$

7. *for all* $C \subseteq A,\ {}^*\,(f\,[C]) = ({}^*f)\,[{}^*C]$.

4.6 Internal elements

In the following, let $* : \widehat{S} \to \widehat{{}^*S}$ be a fixed nonstandard map.

From the transfer principle, we obtain that, $\forall n \in \mathbb{N}$, ${}^*S_n$ is transitive

$$ {}^*S_n \in {}^*S_{n+1} \in \widehat{{}^*S} $$

and, if $A \in {}^*S_{n+1}$ then $A \subseteq {}^*S_n$. Let

$$ St = \left\{ {}^*A \,\middle|\, A \in \widehat{S} \right\} $$

be the set of all internal standard elements.

Definizione 4.6.1 *An element of* $\widehat{{}^*S}$ *is called* ***internal*** *if it belongs to some internal standard set. a set* A *which is not internal is called external. The set of all internal elements of* $\widehat{{}^*S}$

$$ \mathcal{J} = \bigcup St $$

is called the ***internal universe*** *associated with the nonstandard map* $*$.

Observe that all atoms are internal.

Proposizione 4.6.2 *The internal universe is a transitive subset of* $\widehat{{}^*S}$*. Moreover*

$$ St \subseteq \mathcal{J} \subseteq \bigcup_{B \in St} \wp\,(B) $$

and

$$ \mathcal{J} = \bigcup_{n \in \mathbb{N}} {}^*S_n. $$

Proof. Obviously, for all $A \in \widehat{S}$, ${}^*A \subseteq \mathcal{J}$. Hence, for all $n \in \mathbb{N}$, ${}^*S_n \subseteq \mathcal{J}$. If $x \in \widehat{S}$, then $\exists n \in \omega$ such that $x \in S_n$ and hence ${}^*x \in {}^*S_n$ so ${}^*x \in \mathcal{J}$. This proves that $St \subseteq \mathcal{J}$. Now let $x \in {}^*A \in \mathcal{J}$ and observe that $\exists n \in \omega$ such that $A \in S_n$. Hence $x \in {}^*A \in {}^*S_n$ and, by transitivity of S_n, $x \in {}^*S_n$ and $x \subseteq {}^*S_n$, which proves

$$\mathcal{J} \subseteq \bigcup_{n \in \mathbb{N}} {}^*S_n$$

and

$$\mathcal{J} \subseteq \bigcup \{\wp(B) \mid B \in St\}.$$

■

A formula α with constants in $\widehat{{}^*S}$ is said internal if all its constants are internal.

Teorema 4.6.3 (Internal definition principle) *a set $C \in \widehat{{}^*S}$ is internal if and only if can be written in the form*

$$C = \{x \in B \mid \alpha(x, B_1, ..., B_k)\}$$

where B is an internal set and α is a closed internal formula with internal parameters $B_1, ..., B_k$ and only variable x.

Proof. The necessity is obvious, because

$$A = \{x \in A \mid x = x\}$$

For the sufficiency, let $n \in \mathbb{N}$ be such that B and all the constants of α belong to ${}^*S_n$. The formula

$$\forall y_1, ..., y_k, y \in S_n \exists z \in S_{n+1} \forall x \in S_n (x \in z \longleftrightarrow (x \in y \wedge \varphi(x, y_1, ..., y_k)))$$

is true, because if $A_1, ..., A_n, A$ are elements of S_n, then, by the comprehension axiom, there exists the set

$$A' = \{x \in A \mid \varphi(x, A_1, ..., A_k)\}.$$

Moreover, by transitivity of S_n, $A' \subseteq S_n$ and $A' \in S_{n+1}$. Now, by transfer,

$$\forall y_1, ..., y_k, y \in {}^*S_m \exists z \in {}^*S_{n+1} \forall x \in {}^*S_n (x \in z \longleftrightarrow (x \in y \wedge \varphi(x, y_1, ..., y_k)))$$

and, in particular, for $B_1, ..., B_k, B$,

$$\exists z \in {}^*S_{n+1} \forall x \in {}^*S_n (x \in z \longleftrightarrow (x \in B \wedge \varphi(x, B_1, ..., B_k))).$$

Now, $B \in {}^*S_n$, hence $B \subseteq {}^*S_{n+1}$ so that a z satisfying the formula above must be

$$\{x \in B \mid \varphi(x, B_1, ..., B_k)\}$$

which thus belongs to ${}^*S_{n+1}$ and to $\mathcal{J}$. ■

Corollario 4.6.4 *a set $C \in \widehat{{}^*S}$ is internal if and only if it can be written in the form*

$$C = \{x \in {}^*B \mid \alpha(x, B_1, ..., B_k)\}$$

where $B \in \widehat{S}$ and α is a bounded formula with only free variable x and internal parameters $B_1, ..., B_k$.

Proof. It follows from the internal definition principle and the fact that every internal set is contained in a internal standard set. ■

Proposizione 4.6.5 *The internal universe $\mathcal{J}$ is closed under*

1. *the usual insiemistic operations: union, intersection, difference, cartesian product*
2. *domain and range of functions*
3. *section of relations, where if φ is an n-ary relation and $1 \leq j \leq n$, its j-th section is the set*

$$\{x \mid \exists y_1, .., y_{j-1}, y_{j+1}.., y_n,\ (y_1, ..., y_{j-1}, x, y_{j+1}, ..., y_n) \in \varphi\}$$

4. *image and inverse image under internal functions*
5. *composition of relations*

Proof.

1. Let $A, B \in \mathcal{J}$, and $n \in \mathbb{N}$ such that $A, B \in {}^*S_n$, then

$$A \cup B = \{x \in {}^*S_n \mid x \in A \vee x \in B\}$$

$$A \cap B = \{x \in A \mid x \in B\}$$

$$A \backslash B = \{x \in A \mid x \notin B\}$$

$$A \times B = \{z \in {}^*(S_n \times S_n) \mid \exists x \in A \exists y \in B\,(z = (x, y))\}$$

$$\bigcup A = \{x \in {}^*S_n \mid \exists y \in A\,(x \in y)\}$$

$$\bigcap A = \{x \in {}^*S_n \mid \forall y \in A\,(x \in y)\}$$

are internal by the internal definition principle.

2. If $f \in {}^*S_n$ then

$$dom(f) = \{x \in {}^*S_n \mid \exists y \in {}^*S_n, \exists z \in f\,((x, y) = z)\}$$

and

$$ran(f) = \{x \in {}^*S_n \mid \exists y \in {}^*S_n, \exists z \in f\,((y, x) = z)\}.$$

3. The proof is very similar to the one of the previous points.

4. If $A, f \in {}^*S_n$ then

$$f[A] = \{y \in ran(f) \mid \exists x \in A, \exists z \in f((x,y) = z)\}$$

and

$$f^{-1}[A] = \{x \in dom(f) \mid \exists y \in A, \exists z \in f((x,y) = z)\}.$$

5. If φ and ψ are binary relations (the case of k-ary relation for k arbitrary is similar), and $\varphi, \psi \in {}^*S_N$, then

$$\psi \circ \varphi = \{(x,z) \in {}^*(S_N \times S_N) \mid \exists y \in {}^*S_N,\ (x,y) \in \varphi,\ (y,z) \in \psi\}.$$

■

Osservazione 4.6.6 *If $A \subseteq \mathcal{J}$, in general $\bigcup A \notin \mathcal{J}$ (but it does if $A \in \mathcal{J}$). Also, if $C \subseteq B \in \mathcal{J}$, in general $C \notin \mathcal{J}$ (but $C \subseteq J$).*

We can apply the internal definition principle to relations, obtaining the following *internal definition principle for relations.*

Proposizione 4.6.7 *An n-ary relation φ is internal if and only if there exist internal entities $B_1, ..., B_n$ and a bounded formula α with free variables $x_1, ..., x_n$ and possibly some internal parameters, such that*

$$\varphi = \{(x_1, ..., x_n) \in B_1 \times .... \times B_n \mid \alpha(x_1, ..., x_m)\}.$$

Proof. It follows from the internal definition principle and the closure of $\mathcal{J}$ under insiemistic operations and sections of relations. ■

Note that, if $x_1, ..., x_n$ are internal, then such are $\{x_1, ..., x_n\}$ and $(x_1, ..., x_n)$, because if $k \in \mathbb{N}$ is such that $x_1, ..., x_n \in {}^*S_k$ then

$$\{x_1, ..., x_n\} = \{y \in {}^*S_k \mid y = x_1 \vee ... \vee y = x_n\}$$

and

$$(x_1, ..., x_n) \in {}^*S_k \times ... \times {}^*S_k = {}^*(S_k \times ... \times S_k).$$

In particular, since elements of internal entities are internal, an external subset of an internal set must be infinite.

Teorema 4.6.8 *If $A, B \in \widehat{S}$, then*

$$^*\wp(A) = \{M \in \wp(^*A) \mid M \text{ is internal}\}$$

$$^*\left(B^A\right) = \left\{f \in (^*B)^{(^*A)} \mid A \text{ is internal}\right\}$$

Proof. Let $C = \wp(A)$, $D = B^A$ and $n \in \mathbb{N}$ such that $A, B, C, D \in S_n$, hence ${}^*A, {}^*B, {}^*C, {}^*D \in {}^*S_n$. Observe that, for all $m \geq n$,

$$\forall x \in S_m \,(x \subseteq A \longleftrightarrow x \in C)$$

and

$$\forall x \in S_m \,(x : A \to B \longleftrightarrow x \in D)$$

hence

$$\forall x \in {}^*S_m \,(x \subseteq {}^*A \longleftrightarrow x \in {}^*C)$$

and

$$\forall x \in {}^*S_m \,(x : {}^*A \to {}^*B \longleftrightarrow x \in {}^*D)\,.$$

Now, since ${}^*C, {}^*D \subseteq {}^*S_n$, this proves that ${}^*C \subseteq \{M \subseteq {}^*A \mid M \text{ is internal}\}$ and ${}^*D \subseteq \{f : {}^*A \to {}^*B \mid f \text{ is internal}\}$. For the converse, let $M \subseteq A$ be internal and $m \in \mathbb{N}$, $m \geq n$, such that $M \in S_m$, so we can apply the previous formula obtaining $M \in {}^*C$. Analogously for B^A. ■

Proposizione 4.6.9 *If A is an internal set, the family $\wp_I(A)$ of internal subsets of A is internal.*

Proof. Let $n \in \mathbb{N}$ be such that $A \in {}^*S_n$. We have that

$$\begin{aligned} \wp_I(A) &= \{x \in \wp({}^*S_n) \mid x \text{ is internal, } x \subseteq A\} \\ &= \{x \in {}^*\wp(S_n) \mid x \subseteq A\} \end{aligned}$$

is internal by internal definition principle. ■

Proposizione 4.6.10 *If A, B are internal entities, the set $\mathcal{F}$ af all internal functions with domain A and range contained in B and the set $\mathcal{G}$ of internal functions with domain a (necessarily internal) subset of A and range contained in B, are internal.*

Proof. If $n \in \mathbb{N}$ is such that $A, B \in {}^*S_n$ and $A \times B \in {}^*S_n$, we have

$$\begin{aligned} \mathcal{F} &= \{f \subseteq {}^*S_n \mid f \text{ is internal, } f : A \to B, \} \\ &= \{f \in {}^*\wp(S_n) \mid f : A \to B\} \end{aligned}$$

and

$$\mathcal{G} = \{f \in {}^*\wp(S_n) \mid \exists x \in {}^*\wp(S_n)\,,\, x \subseteq A,\ f : x \to B\}$$

are internal by the internal definition principle. ■

4.7 External entities

Teorema 4.7.1 *Let $* : \widehat{S} \to \widehat{{}^*S}$ be a map satisfying the transfer principle, with S infinite. We have that $*$ is a nonstandard map if and only if ${}^\sigma B \neq {}^*B$ for some countably infinite $B \in \widehat{S}$ and, in this case, the following properties hold*

1. *for all $A \in \widehat{S}$ infinite, ${}^{\sigma}A$ is external*

2. *for $A \in \widehat{S}$ infinite, ${}^{\sigma}\wp(A) \subsetneq {}^{*}\wp(A) \subsetneq \wp({}^{*}A)$*

3. *${}^{*}S \backslash S$ is nonempty and contains elements that are internal but not internal standard.*

Proof. The necessity of the condition is clear. For the sufficiency, it is enough to prove that it implies the first point. If, by contradiction, ${}^{\sigma}B$ is internal, then also $C = {}^{*}B \backslash {}^{\sigma}B$ is internal. If $\langle b_n \rangle_{n \in \mathbb{N}}$ is an enumeration of the elements of B, consider the well order relation $\leq$ on B induced by this enumeration. We have

$$\forall x \in \wp(B)\, \exists y \in x\, (\forall z \in x,\, y \leq z)$$

hence

$$\forall x \in {}^{*}\wp(B)\, \exists y \in x\, (\forall z \in x,\, y^{*} \leq z)$$

so that, in particular, when $x = C \in {}^{*}\wp(B)$, we have a ${}^{*} \leq$-minimal element y in C. Now, for all $n \in \mathbb{N}$, we have

$$\forall z \in B\, (z \neq b_0 \wedge ... \wedge z \neq b_n \rightarrow b_{n+1} \leq z)$$

hence, by transfer, since $y \neq {}^{*}b_i$ for all $i \in \mathbb{N}$ (recall that $y \in C = {}^{*}B \backslash {}^{\sigma}B$), we have that $\forall n \in \mathbb{N}$, ${b_n}^{*} \leq y$. Now consider the function $\varphi : B \rightarrow B$ defined by

$$\varphi : b_n \mapsto \begin{cases} b_{n-1} \text{ if } n \geq 1 \\ b_0 \text{ if } n = 0 \end{cases}$$

and observe that

$$\forall x \in B\, (x \neq b_0 \rightarrow x \nleq p(x))$$

hence, by transfer,

$$y^{*} \nleq ({}^{*}p)(y)$$

Now, it is enough to prove that $({}^{*}p)(y) \in C$. We have

$${}^{*}p : {}^{*}B \rightarrow {}^{*}B$$

hence $({}^{*}p)(y) \in {}^{*}B$. Now, if $({}^{*}p)(y) = {}^{*}b_n$ for some $n \in \mathbb{N}$ we have

$$\forall z \in B\, (p(z) = b_n \rightarrow z = b_{n+1} \vee z = b_0)$$

hence, by transfer, $y = {}^{*}b_n$ or $y = {}^{*}b_0$, which can not be. Now, ${}^{\sigma}B$ also cannot be internal, otherwise C would be internal as difference of two internal entities. If now A is another infinite set in $\widehat{S}$, let ψ be a function from A onto B and observe that

$$({}^{*}f)[{}^{\sigma}A] = {}^{\sigma}B$$

since

$$\begin{aligned} {}^{\sigma}B &= \{{}^{*}b \mid b \in B\} \\ &= \{{}^{*}(f(a)) \mid a \in A\} \\ &= \{({}^{*}f)({}^{*}a) \mid a \in A\} \\ &= ({}^{*}f)[{}^{\sigma}A]. \end{aligned}$$

Thus, if ${}^{\sigma}A$ is internal, then ${}^{\sigma}B$ is internal too, as image of an internal set through a internal standard (hence, internal) function. As for the second point, the first strict inclusion follows applying point one to $\wp(A)$ and the second too follows from point one and from the fact that ${}^*\wp(A) = \wp({}^*A) \cap \mathcal{J}$. As for the third part, ${}^*S \backslash S$ is nonempty from the first point and from the fact that ${}^*a = a$ for all $a \in S$. Moreover, if $b \in {}^*S \backslash S$ then b is not internal standard, otherwise there would be $c \in S$ such that

$$b = {}^*c = c.$$

■

4.8 Nonstandard real analysis

In the following we will assume that a nonstandard map $* : \widehat{S} \to \widehat{T}$ has been chose, with $\mathbb{R} \in \widehat{S}$.

It can be easily proven, by transfer, that ${}^*\mathbb{R}$, endowed with the operations ${}^*+$ and ${}^*\cdot$ and the order relation ${}^*\leq$, is an ordered field such that every upper bounded internal subset has a least upper bound. Also the $*$ map restricted to $\mathbb{R}$ is an embedding of ordered field with image ${}^{\sigma}\mathbb{R}$. In the following we will identify $\mathbb{R}$ and its isomorphic copy ${}^{\sigma}\mathbb{R}$. Moreover, ${}^*\mathbb{N}$ is an ordered additive sub semigroup of ${}^*\mathbb{R}$ such that every internal subset has a minimum element. Since $*$ is a nonstandard map, we have $\mathbb{R} \subsetneq {}^*\mathbb{R}$ and $\mathbb{N} \subsetneq {}^*\mathbb{N}$.

The elements of ${}^*\mathbb{R}$ are called hyperreal numbers. We say that an hyperreal number x is

- **finite** if there is $n \in \mathbb{N}$ such that $|x| \leq n$
- **infinite** if it is not finite
- **infinitesimal** if x^{-1} is infinite

The set of finite and infinitesimal numbers are denoted by $Fin({}^*\mathbb{R})$ and $o({}^*\mathbb{R})$ respectively. We set also $\mathbb{N}_\infty = {}^*\mathbb{N} \backslash \mathbb{N}$ and $\mathbb{R}_\infty = {}^*\mathbb{R} \backslash Fin({}^*\mathbb{R})$.

Below there are some obvious facts about finite, infinite and infinitesimal numbers:

- $Fin({}^*\mathbb{R})$ is a convex subring of ${}^*\mathbb{R}$
- $x \in \mathbb{R}_\infty$ iff, $\forall n \in \mathbb{N}$, $|x| > n$, iff, $\forall y \in Fin(\mathbb{R})_+$, $|x| > y$ iff $\exists z \in {}^*\mathbb{R}$, z infinite and $|x| > |z|$ iff $\exists n \in \mathbb{N}_\infty$, $|x| > n$, iff $\frac{1}{x}$ is infinitesimal
- $x \in o({}^*\mathbb{R})$ iff, $\forall n \in \mathbb{N}$, $|x| < \frac{1}{n}$, iff, $\forall y \in Fin({}^*\mathbb{R})_+$, $|x| < y$

Observe that $Fin({}^*\mathbb{R})$, $o({}^*\mathbb{R})$ $\mathbb{R}$, $\mathbb{N}$, $\mathbb{R}_\infty \cap {}^*\mathbb{R}_-$ are external, because they are upper bounded in ${}^*\mathbb{R}$ but have no least upper bound, while $\mathbb{R}_\infty \cap {}^*\mathbb{R}_+$ and $\mathbb{N}_\infty$ are external because they are lower bounded in ${}^*\mathbb{R}$ but have not greatest lower bound. Thus, $\mathbb{R}_\infty$ is external too, otherwise $\mathbb{R}_\infty \cap {}^*\mathbb{R}_+$ would be internal.

We say that two hyperreal numbers x, y are **infinitely close** and we write $x \approx y$ if $x - y \in o\left({}^*\mathbb{R}\right)$. It turns out that $\approx$ is an equivalence relation. Moreover, for all $x_1, x_2, y_1, y_2 \in {}^*\mathbb{R}$ such that $x_1 \approx y_1$ and $x_2 \approx y_2$, we have

1. $x_1 \pm x_2 \approx y_1 \pm y_2$
2. $x_1 x_2 \approx y_1 y_2$ if x_1, x_2 are finite
3. $\frac{x_1}{x_2} \approx \frac{y_1}{y_2}$ if x_1 is finite and x_2 is not infinitesimal

Every finite hyperreal number x is infinitely close to one and only one standard real number, which is called its **standard part** $st\left(x\right)$. This fact is easily proven: if we set

$$A = \{y \in \mathbb{R} \mid y < x\}$$

then A is an upper bounded subset of $\mathbb{R}$ and so, by the completeness of $\mathbb{R}$, it has an upper bound (in $\mathbb{R}$) which must be infinitely close to x.

It is easily seen that $st : Fin\left({}^*\mathbb{R}\right) \to \mathbb{R}$ is a (weakly) order preserving epimorphism whose kernel is $o\left({}^*\mathbb{R}\right)$. Moreover, st is external, as such is its domain $Fin\left({}^*\mathbb{R}\right)$. For every $x \in {}^*\mathbb{R}$, we denote the set of hyperreal numbers infinitely close to x by $mon\left(x\right)$ and call it the **monad** of x. It is easily seen that $mon\left(0\right) = o\left({}^*\mathbb{R}\right)$ and, $\forall x \in Fin\left({}^*\mathbb{R}\right)$, $mon\left(x\right) = x + o\left({}^*\mathbb{R}\right)$. The monads are the equivalence classes of the equivalence relation $\approx$, hence they form a partition of ${}^*\mathbb{R}$.

One of the most important facts in nonstandard analysis is the so called **permanence principle**. If $\alpha\left(x\right)$ is a predicate in the only free variable x with possibly some internal parameters, the following two facts

1. $\exists n_0 \in \mathbb{N}$, $\forall n \in \mathbb{N}$, if $n \geq n_0$ then $\alpha\left(n\right)$
2. $\forall \nu \in \mathbb{N}_\infty$, $\alpha\left(\nu\right)$

In fact, suppose that $n_0 \in \mathbb{N}$ is such that

$$\forall n \in \mathbb{N}, \text{ if } n \geq n_0, \text{ then } \alpha\left(n\right)$$

By transfer, we obtain

$$\forall n \in {}^*\mathbb{N}, \text{ if } n \geq n_0, \text{ then } \alpha\left(n\right)$$

and, in particular,

$$\forall n \in {}^*\mathbb{N}, \text{ if } n \geq n_0, \text{ then } \alpha\left(n\right)$$

For the converse, suppose by contradiction that,

$$\forall n_0 \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq n_0 \text{ and } \neg\alpha\left(n\right)$$

By transfer, we obtain

$$\forall n_0 \in {}^*\mathbb{N}, \exists n \in {}^*\mathbb{N}, n \geq n_0 \text{ and } \neg\alpha\left(n\right)$$

and, in particular, taking $n_0 \in \mathbb{N}_\infty$, we get $n \in \mathbb{N}_\infty$ such that $\neg\alpha(n)$, contradicting 2.

The implication $1 \Rightarrow 2$ is commonly referred to as **overspill principle**, while the converse implication is referred to as **underspill principle**.

Applying the permanence principle to the formula

$$\beta(n) \equiv \exists m \in \mathbb{N},\ m \geq n \text{ and } \alpha(m)$$

we deduce that the following two statements are equivalent:

1. $\forall n_0 \in \mathbb{N}$, $\exists n \in \mathbb{N}$, $n \geq n_0$ and $\alpha(n)$
2. $\exists \nu \in \mathbb{N}_\infty$, $\alpha(\nu)$

It is customary to call also these implications overspill and underspill principles.

A similar principle, the *Cauchy permanence principle*, holds for ${}^*\mathbb{R}$: if $\alpha(x)$ is a predicate in the only free variable x with possibly internal parameters, the statements

1. $\exists x_0 \in \mathbb{R}$, $\forall x \in \mathbb{R}$, if $x \geq x_0$ then $\alpha(x)$
2. $\exists \xi \in {}^*\mathbb{R}$ positive infinite such that $\alpha(\xi)$

are equivalent, as well as the statements,

1. $\forall x_0 \in \mathbb{R}$, $\exists x \in \mathbb{R}$, $x \geq x_0$ and $\alpha(x)$
2. $\exists \xi \in {}^*\mathbb{R}$ positive infinite such that $\alpha(\xi)$

are equivalent. The proof is the similar to the one of the overspill principle.

From the Cauchy permanence principle, considering the formula $\beta(x) = \alpha\left(\frac{1}{x}\right)$ we obtain also the equivalence of

1. $\exists x_0 \in \mathbb{R}_+$, $\forall x \in \mathbb{R}_+$, if $0 < x < x_0$ then $\alpha(x)$
2. $\forall \varepsilon \in {}^*\mathbb{R}$ positive infinitesimal, $\alpha(\varepsilon)$

and of

1. $\forall x_0 \in \mathbb{R}_+$, $\exists x \in \mathbb{R}_+$ such that $0 < x < x_0$ and $\alpha(x)$
2. $\exists \varepsilon \in {}^*\mathbb{R}$ positive infinitesimal such that $\alpha(\varepsilon)$

4.9 Enlargement and saturation

Throughout this section we assume that $* : \widehat{S} \to \widehat{{}^*S}$ is a nonstandard map, with S infinite.

Definizione 4.9.1 *Let k be a cardinal number. The elementary embedding $*$ is called*

- *a k-**enlargement** if, for every set $\mathcal{A}$ of entities of $\widehat{S}$ of cardinality $< k$ with the f.i.p., we have $\bigcap {}^\sigma\mathcal{A} \neq \varnothing$*

- *k-**saturated** if, for every set $\mathcal{B}$ of internal entities of $\widehat{{}^*S}$ of cardinality $< k$ with the f.i.p., we have $\bigcap \mathcal{B} \neq \varnothing$.*

We remark that it is customary to call an $\aleph_1$-saturated nonstandard map, **countably saturated**.

We can give an equivalent characterization of k-enlargements and k-saturated maps, in terms of satisfaction of relations.

If φ is a binary relation, we say that φ is **satisfied** by $b \in ran\,(\varphi)$ on $A \subseteq dom\,(\varphi)$ if $A \times \{b\} \subseteq \varphi$. We call φ **concurrent** on $A \subseteq dom\,(\varphi)$ if, for all $A_0 \subseteq A$ finite, $\exists b \in ran\,(\varphi)$ such that φ is satisfied by b on A_0.

Teorema 4.9.2 *If k is a cardinal number, then the following statements are equivalent*

- *$*$ is a k-enlargement if and only if, for every binary relation $\varphi \in \widehat{S}$ with cardinality $< k$, if φ is concurrent on $A \subseteq dom\,(\varphi)$, then there is $b \in ran\,({}^*\varphi)$ that satisfies ${}^*\varphi$ on ${}^\sigma A$*

- *$*$ is k-saturated if and only if for every (non necessarily internal) binary relation φ of cardinality $< k$ and for all (non necessarily internal) $A \subseteq dom\,(\varphi)$, such that, $\forall a \in A$, $\varphi\,[a]$ is internal, if φ is concurrent on A, then φ is satisfied on A.*

For a proof, see [**?**].

If $*$ is a nonstandard extension, I is an infinite set and $\iota \in {}^*I \backslash I$, then

$$\{A \subset I \mid \iota \in {}^*A\}$$

is a nonprincipal ultrafilter over I. If $*$ is a $\left(2^{|I|}\right)^+$-enlargement, then every nonprincipal ultrafilter $\mathcal{U}$ over I is of this form. In fact, if $\iota \in \bigcap {}^\sigma\mathcal{U}$, then $\mathcal{U} = \mathcal{U}_\iota$.

Chapter 5

Sofic and hyperlinear groups

5.1 Standard definition and characterization

A bounded bi-invariant metric groups is a group which is also a metric space of diameter ≤ 2 such that the multiplication and inverse functions are continuous, and the metric is invariant with respect to right and left multiplication.

If $n \in \mathbb{N}$, denote by S_n the permutation groups of n and by $U_n \subset \mathbb{M}_n$ the group of unitary $n \times n$ matrices over $\mathbb{C}$. If $\sigma, \tau \in S_n$, define

$$d(\sigma, \tau) = \frac{|\{i \in n \mid \sigma(i) \neq \tau(i)\}|}{n}.$$

If $A, B \in U_n$, define

$$\begin{aligned} d(A, B) &= \sqrt{\frac{\sum_{1 \leq i,j \leq n} |A_{i,j} - B_{i,j}|^2}{n}} \\ &= \sqrt{\frac{tr\left((A-B)^*(A-B)\right)}{n}} \\ &= \frac{\|A - B\|_2}{\sqrt{n}} \end{aligned}$$

It is easily checked that S_n and U_n are bounded bi-invariant metric groups of diameter 1 and 2 respectively.

The function from S_n to U_n sending σ to the element A_σ of U_n defined by

$$A_\sigma(e_i) = e_{\sigma(i)}$$

if $\{e_0, .., e_{n-1}\}$ is the canonical base of $\mathbb{C}^n$, is a homomorphism such that

$$\begin{aligned} d(\sigma,\tau) &= \frac{|\{i \in n \mid \sigma(i) \neq \tau(i)\}|}{n} \\ &= \frac{\left|\left\{i \in n \mid \left(\tau^{-1}\sigma\right)(i) \neq i\right\}\right|}{n}\grave{u} \\ &= \frac{tr\left(\mathbb{1} - A_{\tau^{-1}\sigma}\right)}{n} \\ &= \frac{tr\left(\left(\mathbb{1} - A_{\tau^{-1}\sigma}\right) + \left(\mathbb{1} - A_{\sigma\tau^{-1}}\right)\right)}{2n} \\ &= \frac{tr\left(\left(A_\sigma - A_\tau\right)^*\left(A_\sigma - A_\tau\right)\right)}{2n} \\ &= \frac{1}{2} d\left(A_\sigma, A_\tau\right)^2 \end{aligned}$$

If G is a discrete group, Γ is a metric group, $\varepsilon > 0$ and $F \subset G$ a finite subset of G, a (F,ε)-almost embedding of G into Γ is a function $\varphi : F \to \Gamma$ such that

- for every $g, h \in F$, if $gh \in F$,

$$d(\varphi(gh), \varphi(g)\varphi(h)) < \varepsilon$$

- if $e_G \in F$,

$$d(\varphi(e_G), e_\Gamma) < \varepsilon$$

- for every $g, h \in F$ distinct,

$$d(\varphi(g), \varphi(h)) \geq \frac{1}{2} diam(\Gamma)$$

Definizione 5.1.1 *A discrete group G is called sofic (resp. hyperlinear) if, for every $F \subset G$ finite and $\varepsilon > 0$, there is $n \in \mathbb{N}$ and an (F,ε)-almost embedding $\varphi : F \to S_n$ (resp. $\varphi : F \to U_n$)*

Observe that every sofic groups is hyperlinear. It is not known if the reverse implication holds.

For example, every residually finite group is sofic. Remember that a residually finite groups is a group that admits a separating family of homomorphism into finite groups or, equivalently, into finite permutation groups. In fact, suppose G is residually finite and $F \subset G$ is finite. If $\alpha \in (0,1)$, there is $N \in \mathbb{N}$ such that, for every $g, h \in F$ with $g \neq h$, there is an embedding $\varphi_{g,h} : G \to S_N$ such that $d\left(\varphi_{g,h}(g), \varphi_{g,h}(h)\right) \geq \frac{1}{2}$. Define now,

$$\Phi : G \to \prod_{(g,h) \in F, g \neq h} S_N \simeq S_{|F|(|F|-1)N}$$

by

$$\Phi(x)(g,h) = \varphi_{g,h}(x).$$

Observe that Φ is a homomorphism and, if $g \neq h \in F$,

$$d\left(\Phi\left(g\right),\Phi\left(h\right)\right) \geq d\left(\varphi_{g,h}\left(g\right),\varphi_{g,h}\left(h\right)\right) \geq \frac{1}{2}$$

5.2 Nonstandard characterization

If $\left(\Gamma_n\right)_{n\in\mathbb{N}}$ is a sequence of bounded bi-invariant metric groups and $\mathcal{U}$ is a ultrafilter over $\mathbb{N}$, define $\prod_n{}^{\mathcal{U}}\Gamma_n$ as the quotient of $\prod_n \Gamma_n$ with metric

$$d\left(\mathbf{a},\mathbf{b}\right) = \lim_n d_n\left(a_n, b_n\right)$$

with respect to the normal subgroup

$$\left\{\mathbf{a} \in \prod_n \Gamma_n \,|\, d\left(\mathbf{a},\mathbf{e}\right) = 0\right\}.$$

If $*$ is a nonstandard extension and $\nu \in {}^*\mathbb{N}$, define Γ_ν as the value at ν of the nonstandard extension of the sequence $\left(\Gamma_n\right)_{n\in\mathbb{N}}$ and $\widehat{\Gamma}_\nu$ as the quotient of Γ_n with respect to the normal subgroup

$$\left\{g \in \Gamma_\nu \,|d\left(g,0\right) \approx 0\right\}$$

Lemma 5.2.1 *If $\nu \in {}^*\mathbb{N}$ and $\mathcal{U}_\nu = \left\{A \subset \mathbb{N} \,|\, \nu \in {}^*A\right\}$, then $\prod_n^{\mathcal{U}_\nu} \Gamma_n$ can be embedded in $\widehat{\Gamma}_\nu$*

Proof. Define the function Ψ from $\prod_n^{\mathcal{U}_\nu} \Gamma_n$ to $\widehat{\Gamma}_\nu$ sending $\left[\left(g_n\right)_{n\in\mathbb{N}}\right]$ to $\left[g_\nu\right]$. Observe that Ψ is well defined and one to one. In fact, $\left[\left(g_n\right)\right] = \left[\left(g'_n\right)\right]$ iff, $\forall\varepsilon > 0$, $\left\{n \in \mathbb{N} \,|\, d\left(g_n, g'_n\right) < \varepsilon\right\} \in \mathcal{U}$ iff, $\forall\varepsilon > 0$, $d\left(g_\nu, g'_\nu\right) < \varepsilon$ iff $d\left(g_\nu, g'_\nu\right) \approx 0$ iff $\left[g_\nu\right] = \left[g'_\nu\right]$. Clearly, Ψ is a homomorphism. ∎

It is clear that a group G is sofic (resp. hyperlinear) iff every finitely generated subgroup of G is sofic (resp. hyperlinear). Thus, there is no loss of generality in considering only countable groups.

Teorema 5.2.2 *If G is a countable discrete group and $*$ is a $\mathfrak{c}^+$-enlargement, the following statements are equivalent*

1. *G is sofic*
2. *G can be embedded in $\widehat{S}_\nu$ for some $\nu \in \mathbb{N}_\infty$*
3. *G can be embedded in $\widehat{S}_\nu$ for every $\nu \in \mathbb{N}_\infty$*
4. *G can be embedded in $\prod_n^{\mathcal{U}} S_n$ for some nonprincipal ultrafilter $\mathcal{U}$*
5. *G can be embedded in $\prod_n^{\mathcal{U}} S_n$ for every nonprincipal ultrafilter $\mathcal{U}$*

Proof.

$1 \Rightarrow 5$ Consider, a monotone decreasing vanishing sequence (ε_n) of positive real numbers, an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of G such that $\bigcup_n F_n = G$ and functions $\varphi_n : G \to S_n$ such that φ_n is an (F_n, ε_n)-almost homomorphism. If $\mathcal{U}$ is a nonprincipal ultrafilter over $\mathbb{N}$, define

$$\Phi : G \to \prod_n{}^{\mathcal{U}} S_n$$

by

$$\Phi(g) = (\varphi_n(g))_{n\in\mathbb{N}}$$

If $g, h \in G$ and $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\varepsilon_N < \varepsilon$ and $\{e, g, h\} \subset F_N$. Thus, for every $n \geq N$,

$$d(\varphi_n(gh), \varphi_n(g)\varphi_n(h)) < \varepsilon$$

and

$$d(\varphi_n(e), e) < \varepsilon$$

and, if $g \neq h$,

$$d(\varphi_n(g), \varphi_n(h)) \geq \frac{1}{2}.$$

Thus,

$$d(\Phi(gh), \Phi(g)\Phi(h)) < \varepsilon$$

and

$$d(\Phi(e), e) < \varepsilon$$

Moreover, if $g \neq h$, then

$$d(\Phi(g), \Phi(h)) \geq \frac{1}{2}$$

Since this is true for every $\varepsilon > 0$,

$$\Phi(gh) = \Phi(g)\Phi(h)$$

and

$$\Phi(e) = e$$

$5 \Rightarrow 4$ Obvious

$5 \Rightarrow 3$ If $\nu \in \mathbb{N}_\infty$, consider the ultrafilter $\mathcal{U}_\nu$. Thus, G can be embedded in $\prod_n^{\mathcal{U}_\nu} S_n$ and, by the previous lemma, $\prod_n^{\mathcal{U}_\nu} S_n$ can be embedded in $\widehat{S}_\nu$

$3 \Rightarrow 2$ Obvious

$2 \Rightarrow 1$ Suppose F is a finite subset of G and $\varepsilon > 0$. Consider an embedding $\widehat{\Phi} : G \to \widehat{S}_\nu$, which induces a function $\Phi : G \to S_\nu$ such that, for every $g, h \in F$, $d(\Phi(gh), \Phi(g)\Phi(h)) \approx 0$, $d(\Phi(e), e) \approx 0$ and, for every $g, h \in F$, $g \neq h$, $d(\Phi(g), \Phi(h))$ is not infinitesimal. Define

$$\eta = \inf\{d(\Phi(g), \Phi(h)) \mid g, h \in F, g \neq h\}$$

and observe that η is a positive non-infinitesimal hyperreal number. Pick $N \in \mathbb{N}$ such that $(1-\eta)^N \leq \frac{1}{2}$ and consider the function $\Psi : G \to S_\nu^N \simeq S_{\nu^N}$ defined by

$$\Psi(g) = (\Phi(g), ..., \Phi(g)).$$

If $g, h \in G$, then

$$d(\Psi(g), \Psi(h)) = 1 - (1 - d(\Phi(g), \Phi(h)))^N.$$

Thus, if $g, h \in F$, then

$$d(\Psi(gh), \Psi(g)\Psi(h)) \approx 0$$

$$d(\Psi(e), e) \approx 0$$

and, if $g \neq h$, then

$$d(\Psi(g), \Psi(h)) = 1 - (1 - d(\Phi(g), \Phi(h)))^N \geq 1 - (1-\eta)^N \geq \frac{1}{2}.$$

Consider thus the formula $\exists \nu \in {}^*\mathbb{N}$, $\exists f : F \to S_\nu$, such that, for every $g, h \in F$, $d(f(gh), f(g)f(h)) < \varepsilon$, $d(f(e), e) < \varepsilon$ and, if $g \neq h$, $d(f(g), f(h)) \geq \frac{1}{2}$. By transfer, one gets, $\exists n \in \mathbb{N}$, $\exists f : F \to S_n$ such that, for every $g, h \in F$, $d(f(gh), f(g)f(h)) < \varepsilon$, $d(f(e), e) < \varepsilon$ and, if $g \neq h$, $d(f(g), f(h)) \geq \frac{1}{2}$. Since this is true for every $\varepsilon > 0$ and $F \subset G$ finite, G is sofic.

■

In the same way, it is proved the following

Teorema 5.2.3 *If G is a countable discrete group and $*$ is a $\mathfrak{c}^+$-enlargement, the following statements are equivalent*

1. *G is hyperlinear*
2. *G can be embedded in $\widehat{U}_\nu$ for some $\nu \in \mathbb{N}_\infty$*
3. *G can be embedded in $\widehat{U}_\nu$ for every $\nu \in \mathbb{N}_\infty$*
4. *G can be embedded in $\prod_n^{\mathcal{U}} U_n$ for some nonprincipal ultrafilter $\mathcal{U}$*
5. *G can be embedded in $\prod_n^{\mathcal{U}} U_n$ for every nonprincipal ultrafilter $\mathcal{U}$*

These theorems justify the following

Definizione 5.2.4 *If $\mathcal{U}$ is a nonprincipal ultrafilter over $\mathbb{N}$, the ultraproduct $\prod_n^{\mathcal{U}} S_n$ (resp. $\prod_n^{\mathcal{U}} U_n$) is called a universal sofic (resp. hyperlinear) group.*

Part III

Continuum logic and the order property

Chapter 6

Logic for metric structures

6.1 Languages for operator algebra

A **language** $\mathcal{L}$ consists of

- a set $\mathcal{S}$ of **sorts**, whose elements are meant to represent spaces. To every $S \in \mathcal{S}$, it is associated a symbol d_S and directed set $\mathcal{D}_S$ of **domains**. For every $D \in \mathcal{D}_S$ a natural number K_D is given

- **sorted function symbols** $f : S_1 \times ... \times S_n \to S$ together with, for every choice of domains $D_i \in \mathcal{D}_{S_i}$ for $i \in \{1, 2, ..., n\}$, and every $j \in \{1, 2, ..., n\}$, a domain $D_f^{D_1,...,D_n} \in \mathcal{D}_S$ and a real valued function of real variable $\delta_{f,j}^{D_1,...,D_n}$ vanishing in 0. A zerary sorted function symbol stands for a sorted constant symbol.

- **sorted relation symbols** R on $S_1 \times .... \times S_n$ together with, for $j \in \{1, 2, ..., n\}$ and every choice of $D_i \in \mathcal{D}_{S_i}$ for $i \in \{1, 2, ..., n\}$, a real valued function of real variable $\delta_{R,j}^{D_1,...,D_n}$ and a positive real number $N_R^{D_1,...,D_n}$

An $\mathcal{L}$-structure M is a function that assigns, to every sort S, a metric space $M(S)$ with metric d_S^M and, to every domain D relative to S, a subset $M(D)$ of $M(S)$ complete with respect to d_S of diameter $\leq K_D$, in such a way that, if $D \leq D'$, then $M(D) \subset M(D')$, and the family $\{M(D) : D \in \mathcal{D}_S\}$ is a cover of $M(S)$. Moreover, to every sorted function symbol $f : S_1 \times ... \times S_n \to S$ is associated a function $f^M : M(S_1) \times ... \times M(S_n) \to M(S)$ such that, for every choice of $D_i \in \mathcal{D}_{S_i}$ for $i \in \{1, 2, ..., n\}$, the restriction of f to $M(D_1) \times ....M(D_n)$ is uniformly continuous with modulus of continuity relative to the j-th variable $\delta_{f,j}^{D_1,...,D_n}$ and has image contained in $M\left(D_f^{D_1,..,D_n}\right)$. Finally, to every sorted relation symbol R it is a associated a real valued function R on $M(S_1) \times .... \times M(S_n)$ such that, for every choice of $D_i \in \mathcal{D}_{S_i}$, $i \in \{1, 2, .., n\}$, the restriction of R to $M(D_1) \times ... \times M(D_n)$ is uniformly continuous with modulus of continuity $\delta_{R,j}^{D_1,\dots,D_n}$ relative to the j-th variable, and bounded in absolute value by $N_R^{D_1,\dots,D_n}$.

In the following, I will use the following notation. If $\mathcal{L}$ is a language, a multisort $\overline{S}$ indicates a finite sequence $(S_1, ..., S_n)$ of sorts of $\mathcal{L}$. A multi-domain $\overline{D}$ of multi-sort $\overline{S}$ is a finite sequence $(D_1, ..., D_n)$ of domains such that $D_i \in \mathcal{D}_{S_i}$ for every i.

For example, the language $\mathcal{L}_{C^*}$ of C*-algebras consists of two sorts U (for the C*-algebra itself) and $\mathbb{C}$ (to represent a copy of the complex numbers). The domains relative to U are $\{D_n\}_{n\in\mathbb{N}}$, to be seen as the balls of radius $n \in \mathbb{N}$, and the domains for $\mathbb{C}$ are $\{B_n\}_{n\in\mathbb{N}}$, to be seen as the discs of radius $n \in \mathbb{N}$. The sorted relation and function symbols are:

- the constant 0 in U and the constants $1, i$ in $\mathbb{C}$
- a binary function symbol $f : \mathbb{C} \times U \to U$, to be interpreted as the multiplication by scalars
- a unitary function symbol $^* : U \to U$, for the involution of U, and $: \mathbb{C} \to \mathbb{C}$, for the complex conjugation
- binary function symbols $+, \cdot : U \times U \to U$ and $+, \cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$

The metric symbols d_U and $d_{\mathbb{C}}$ relative to U and $\mathbb{C}$ should be seen as the norm distance and the Euclidean distance respectively. It is straightforward to write down range domains and moduli of continuity δ associated to the sorted function symbols and the domains.

The language $\mathcal{L}_{vN}^{Tr}$ for tracial von Neumann algebras consists of the sorts U and $\mathbb{C}$, together with domains $\{D_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$, where D_n has to be interpreted as the ball of radius n in operator norm. The metric symbols d_U relative to U has to be seen as the 2-distance induced by the trace. The sorted relation and function symbols are, in addition to the previous ones,

- the constant 1 in U
- a function symbol $\tau : U \to \mathbb{C}$ for the trace
- a unary relation symbol Re on $\mathbb{C}$ for the real part

The range domains and moduli of continuity are easily determined

It is worth observing that the operator norm is not in this case part of the language, and is not even definable, since it is not continuous with respect to the 2-norm.

The language for bounded bi-invariant metric group has only one sort S and a unique domain $D = S$. There is a binary function symbol $\cdot$, for the operation, a unary function symbol j for the inverse and constant symbol e for the multiplicative identity. The metric is bounded by 2 and the uniform continuity module for the operation in each variable and for j is the identity function.

6.2 Formulas, models and theories

We suppose that, for every sort $S \in \mathcal{S}$, we have **variables** of sort S. A variable of sort S is also a **term** of sort S. If $f : S_1 \times .... \times S_n \to S$ is a sorted function symbol and if $t_1, ..., t_n$ are terms of sort $S_1, ..., S_n$, then $f(t_1, ..., t_n)$ is a term of sort S. If R is a sorted relation on $S_1 \times ... \times S_n$ and $t_1, ..., t_n$ are terms of sort $S_1, ..., S_n$, then $R(t_1, ..., t_n)$ is a **basic formula**. If $\varphi_1, ..., \varphi_n$ are formulae and $g : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then $f(\varphi_1, ..., \varphi_n)$ is a formula. Finally, if φ is a formula containing a variable x of sort S and $D \in \mathcal{D}_S$, then $\inf_{x \in D} \varphi(x)$ and $\sup_{x \in D} \varphi(x)$ are formulae. A variable x of sort S in a formula φ is called bounded if it is preceded by $\inf_{x \in D}$ or $\sup_{x \in D}$ for some $D \in \mathcal{D}_S$, free otherwise. A formula without free variables is called a **sentence**.

The interpretation φ^M of a formula φ in a structure M is defined in the obvious way. If φ is a sentence, then φ^M is a real number. The set of sentences φ such that $\varphi^M = 0$ is called the **complete theory** $Th(M)$ of M. Since, for every sentence φ and every real valued continuous function of real variable g, $g(\varphi)$ is a sentence, the complete theory of M determines φ^M for every sentence φ. Two structures M and N are said to be elementarily equivalent if $Th(M) = Th(N)$.

It can be proved by induction on the complexity of a formula φ with free variables $\overline{x} = (x_1, ..., x_n)$ of multi-sort $\overline{S} = (S_1, ..., S_n)$ that, for every multi-domain $\overline{D} = (D_1, ..., D_n)$ relative to $\overline{S}$, there is a positive real number $N_\varphi^{\overline{D}}$ and a real valued function of a real variable $\delta_\varphi^{\overline{D}}$ vanishing in 0 such that, for every $\mathcal{L}$-structure M, the restriction of φ^M to $M(D_1) \times ... \times M(D_n)$ attains values in $\left[-N_\varphi^{\overline{D}}, N_\varphi^{\overline{D}}\right]$ and admits $\delta_\varphi^{\overline{D}}$ as modulus of continuity.

A theory T will be a set of formulae. We say that a structure M is a model of T, and write $M \models T$, iff $T \subset Th(M)$.·

If M is an $\mathcal{L}$-structure and $A \subset M$ define the language $\mathcal{L}(A)$ as the language obtained from $\mathcal{L}$ adding a constant symbol $\widetilde{a}$ of sort S_i in D for every $i \in \{1, 2, ..., n\}$, $D \in \mathcal{D}_{S_i}$ and $a \in A_i \cap M(D)$. A formula φ in the language $\mathcal{L}(A)$ is called a $\mathcal{L}$-formula with parameters from A. Denote by $(M, a)_{a \in A}$ the $\mathcal{L}(A)$-structure obtained by M interpreting $\widetilde{a}$ as a for every $a \in A$.

A map $\Theta : M \to N$ is an elementary embedding if, for all formulae ψ with parameters in M, $\psi^M = \psi^N \circ \Theta$. It can be proved that every isomorphism $\Theta : M \to N$ is an elementary embedding and every elementary embedding $\Theta : M \to N$ is an isomorphism onto its image. None of these implications reverses in general.

6.3 Axiomatizability

A category $\mathcal{C}$ is said to be **axiomatizable** if there is a language $\mathcal{L}$, an $\mathcal{L}$-theory $\mathcal{T}$ and a set of conditions Σ such that the category $\mathcal{C}(\mathcal{T}, \Sigma)$ that has as objects the models of $\mathcal{T}$ and as morphisms the maps between models that preserve all the conditions in Σ, is equivalent to $\mathcal{C}$.

A class of algebras is said axiomatizable is such is the category that has the isomorphism class of algebras as objects and morphisms of algebras as morphisms.

As example of possible choices of Σ, if Σ is the set of all conditions, then the morphisms in $\mathcal{C}(\mathcal{T},\Sigma)$ are exactly the elementary embeddings. If Σ is the set of conditions $\varphi \leq r$ and $r \leq \varphi$ for φ basic formula, then the morphisms in $\mathcal{C}(\mathcal{T},\Sigma)$ are the isomorphisms onto the image, and if Σ is the set of conditions $\varphi \leq r$ for φ basic formula, then the morphisms in $\mathcal{C}(\mathcal{T},\Sigma)$ are the homomorphisms.

Observe that, if τ, σ are terms in a language $\mathcal{L}$ of same sort S, the axiom scheme

$$\sup_{x \in D} d_S\left(\tau(x), \sigma(x)\right)$$

for $D \in \mathcal{D}_S$, forces in any model to be $\tau^M = \sigma^M$ on $M(S)$. Analogously, if φ, ψ are formulae with free variables $x_1, ..., x_n$ of sort $S_1, ..., S_n$, then the axiom scheme

$$\sup_{x_1 \in D_1} \sup_{x_2 \in D_2} ... \sup_{x_n \in D_n} \max\left(0, \left(\psi\left(x_1, .., x_n\right)\right) - \varphi\left(x_1, .., x_n\right)\right)$$

where D_i ranges in $\mathcal{D}_{S_i}$ for every $i \in \{1, 2, ..., n\}$, forces to be $\psi \leq \varphi$ on $M(S_1) \times .... \times M(S_n)$

I will consider now the axioms for C*-algebras, on the language $\mathcal{L}_{C^*}$ previously introduced. It is not difficult to give axioms that ensure that U is interpreted as a C*-algebra. In order to ensure that, $\forall n \in \mathbb{N}$, D_n is interpreted as the n-ball of U, one has to add the following axioms (writing $\|x\|$ for $d_U(x, 0)$):

- $\sup_{x \in D_1} \|x\| \leq 1$ (D_1 is contained in the unit ball)
- for every $n \in \mathbb{N}$, $\sup_{a \in D_n} \inf_{b \in D_1} d_U\left(b, \frac{a}{\|a\| + \frac{1}{n}}\right) = 0$ (the open unit ball is contained in D_1)
- for every $n \in \mathbb{N}$, $\sup_{a \in D_n} \inf_{b \in D_1} d_U\left(\frac{1}{n}a, b\right) =$
- 0 ($\frac{1}{n}D_n \subset D_1$)
- for every $n \in \mathbb{N}$, $\sup_{b \in D_1} \inf_{a \in D_n} d_U\left(\frac{1}{n}a, b\right) = 0$ ($\frac{1}{n}D_n$ is dense in D_1)

The first two of these axioms implies that D_1 is interpreted as the unit ball, and the remaining two implies that $D_n = nD_1$ is interpreted as the n-ball. In order to ensure that $\mathbb{C}$ is interpreted as the field of complex numbers with the Euclidean distance and, for every $n \in \mathbb{N}$, B_n is interpreted as the n-disc of $\mathbb{C}$, to the same axioms we enlisted for U one has to add $i^2 + 1$ and

$$\sup_{\lambda \in B_1} \inf_{\mu \in B_1} \inf_{\nu \in B_2} \min\left\{\left|\lambda + |\mu| 1 + |\nu| i\right|, \left|\lambda - |\mu| 1 + |\nu| i\right|, \left|\lambda + |\mu| 1 - |\nu| i\right|, \left|\lambda - |\mu| 1 - |\nu| i\right|\right\},$$

ensuring that $\{1, i\}$ is a base for $\mathbb{C}$ as a vector space over $\mathbb{R}$.

It is not difficult to check now that there is an equivalence of categories between the category of isomorphism classes of C*-algebras and the category $\mathcal{C}(\mathcal{T}_{C^*}, \Sigma)$, where $\mathcal{T}_{C^*}$ is the theory of C*-algebras we have just defined and Σ

is the set of conditions of form $\varphi \leq r$ for φ formula and r real number (ensuring that morphisms in $\mathcal{C}(\mathcal{T}_{C^*}, \Sigma)$ are *-homomorphisms of C*-algebras).

I'll consider now that axioms for von Neumann algebras. As for C*-algebras, it is not difficult to write axioms that ensure that U is interpreted as a tracial *-algebra (it is worth reminding here that the operator norm is not part of the structure nor even a definable function). The axiom scheme

$$\sup_{a \in D_n} \left(\mathrm{Re} \left(\tau(a^* a) - d_U(a, 0)^2 \right) \right)^2$$

for $n \in \mathbb{N}$ forces

$$\|a\|_2^2 = tr(a^* a),$$

where $\|a\|_2 = d_U(a, 0)$.

In order to ensure that U is a von Neumann algebra and D_n is interpreted as the operator norm unit ball, one has to add the following axioms

- for every $n \in \mathbb{N}$,

$$\sup_{a \in D_n} \sup_{x \in D_1} \max\{0, \|ax\|_2 - n\|x\|_2\}$$

  forcing left multiplication by $a \in D_n$ to be a bounded linear operator on U of norm $\leq n$

- for every $n \in \mathbb{N}$,

$$\sup_{a \in D_n} \inf_{b \in D_1} \inf_{c \in D_n} \inf_{d \in D_n} \left(\|a - bc\|_2 + \|c - d^* d\|_2 + \|b^* b - 1\|_2 \right)$$

  expressing the fact that every $a \in U$ has a polar decomposition in U

- $$\sup_{a \in D_1} \sup_{b \in D_1} \inf_{c \in D_1} \left\| c - \frac{a+b}{2} \right\|_2$$

  ensuring that D_1 is convex

I now claim this theory $\mathcal{T}_{vN}^{Tr}$ defines exactly the category of tracial von Neumann algebras.

In fact, if M is a model of $\mathcal{T}_{vN}^{Tr}$, then $M(U)$ is a pre-Hilbert space with respect to the scalar product $\langle x, y \rangle_\tau = \tau(y^* x)$. Moreover, left multiplication by $a \in D_n$ is a bounded linear operator of norm $\leq n$. Moreover, for every $x, y \in M$,

$$\begin{aligned} \langle a^* x, y \rangle_\tau &= \tau(y^* a^* x) \\ &= \tau\left((ay)^* x\right) \\ &= \langle x, ay \rangle_\tau \end{aligned}$$

which shows that the adjoint of the left multiplication by a is the left multiplication by a^*. Thus, it is defined a faithful *-representation π of $M(U)$ on the

Hilbert completion $L^2(M(U), \tau)$ of $(M(U), \langle\cdot,\cdot\rangle_\tau)$. The restriction of π to D_1 is a homeomorphism with respect to the 2-norm topology on D_1 and the strong topology on $B(L^2(M(U), \tau))$. In fact, if $(z_i)_{i\in I}$ is a net in D_1 converging in 2-norm to 0 then, for every $x \in M(U)$, since $\|x\|_2 = \|x^*\|_2$, if $x \in U_n$,

$$\begin{aligned} \|z_i u\|_2 &= \|u^* z_i^*\|_2 \\ &\leq n\|z_i^*\|_2 \\ &= n\|z_i\|_2 \underset{i\to\infty}{\longrightarrow} 0 \end{aligned}$$

Conversely, if

$$(\pi(z_i))_{i\in I}$$

converges to 0 strongly, evaluating in 1 one gets $\lim_{i\in I}\|z_i\|_2 = 0$. Since D_1 is complete in 2-norm and is mapped homeomorphically onto the norm unit ball of $\pi(M(U))$, it follows that the latter is strongly closed and hence $\pi(M(U))$ (and $M(U)$ as well) is a von Neumann algebra. It remains to show that, $\forall n \in \mathbb{N}$, D_n is interpreted as the operator norm n-ball of $M(U)$. Since we still have the axioms that guarantee that $D_n = nD_1$, it is enough to prove that D_1 is interpreted as the operator norm unit ball. By the first of the additional axioms, D_1 is contained in the operator norm unit ball. By the second of the additional axioms, every unitary element belongs to D_1 and D_1 is convex. By the Russo-Dye theorem, every element of the open operator norm unit ball is convex combination of unitaries. This implies that open operator norm unit ball is contained in D_1. By completeness of D_1, this is enough to conclude.

In order to axiomatize the class of tracial factors, consider the terms

$$\xi(x) = \|x - \tau(x)1\|_2$$

and

$$\eta(x) = \sup_{y\in D_1}\|[x,y]\|_2,$$

where $[x,y] = xy - yx$ is the commutant of x and y. Thus, if one adds to the axioms for von Neumann algebras the axiom

$$\sup_{x\in D_1}\max\{0, (\xi(x) - \eta(x))\}$$

which implies $\xi \leq \eta$ in a model, then one gets an axiomatization of tracial von Neumann factors. In fact, if (M,τ) is a tracial factor and $x \in M$ is an element of operator norm ≤ 1, then, by the Diximier property of the trace in a factor, for every $\varepsilon > 0$ there is a convex combination $\sum_{j=1}^n \lambda_j u_j x u_j^*$ such that

$$\left\|\tau(x)1 - \sum_{j=1}^n \lambda_j u_j x u_j^*\right\| \leq \varepsilon$$

and hence

$$\begin{aligned}\xi(x) &= \|x-\tau(x)1\|_2 \\ &\leq \varepsilon + \left\| x - \sum_{j=1}^{n} \lambda_j u_j x u_j^* \right\|_2 \\ &\leq \varepsilon + \sum_{j=1}^{n} \lambda_j \left\| x - u_j x u_j^* \right\|_2 \\ &= \varepsilon + \sum_{j=1}^{n} \lambda_j \|[x,u_j]\|_2 \\ &\leq \varepsilon + \sup_{y\in D_1} \|[x,y]\|_2 = \eta(x) + \varepsilon\end{aligned}$$

Since this is true for every $\varepsilon > 0$, the thesis is proved Conversely, assume $\xi(x) \leq \eta(x)$ for every x in the operator norm unit ball of a von Neumann algebra M. If M is not a factor, then there is a nontrivial central projection p in M, with $0 < \tau(p) < 1$. For this element, we have $\eta(p) = 0$ and

$$\begin{aligned}\xi(p) &= \|p-\tau(p)1\|_2^{\frac{1}{2}} \\ &= \tau\left(p - 2\tau(p)p + \tau(p)^2 1\right)^{\frac{1}{2}} \\ &= \left(\tau(p) - 2\tau(p)^2 + \tau(p)^2\right)^{\frac{1}{2}} \\ &= \left(\tau(p) - \tau(p)^2\right)^{\frac{1}{2}} > 0 = \eta(p)\end{aligned}$$

contradicting the assumption.

In order to axiomatize II_1 factors, it is enough to require the trace to attain an irrational value on some projection. Fix thus an irrational number $\beta \in (0,1)$ and consider the axiom

$$\inf_{a\in D_1} \max\left(\left\|a^*a - (a^*a)^2\right\|_2, |\tau(a^*a) - \beta|\right)$$

I claim that adding this axiom to the list of axioms for tracial factors gives an axiomatization of II_1 factors. In fact, if M is a II_1 factor, then it has a projection p such that $\tau(p) = d(p) = \beta$. Conversely, if a finite factor is not II_1, then it is of type I_n for some $n \in \mathbb{N}$, i.e. it is isomorphic to $\mathbb{M}_n$, and every nontrivial projection in $\mathbb{M}_n$ has dimension $\frac{k}{n}$ for $k \in \{1, ..., n-1\}$.

The axioms for bounded bi-invariant metric groups are the usual axioms for groups, with the addition of

$$\sup_{x,y,z\in S} \left(|d(xz,yz) - d(x,y)| + |d(zx,zy) - d(x,y)|\right)$$

for the by-invariance of the metric.

6.4 Metric ultraproducts

Suppose $\mathcal{L}$ is a language as defined above, $(M_i)_{i\in I}$ is a sequence of $\mathcal{L}$-structures and $\mathcal{U}$ is a nonprincipal ultrafilter on I. The ultraproduct $M = \prod_{i\in I}^{\mathcal{U}} M_i$ will be a structure over the same language $\mathcal{L}$. For every sort $S \in \mathcal{S}$ of $\mathcal{L}$, consider

$$\widetilde{X}_S^{\mathcal{U}} = \left\{ (a_i)_{i\in I} \in \prod_{i\in I} M_i(S) \,|\, \text{for some } D \in \mathcal{D}_S, \text{ for } \mathcal{U}\text{-a.a. } i \in I,\, a_i \in M_i(D) \right\}$$

awith the pseudometric

$$\widetilde{d}_S\left((a_i)_{i\in I}, (b_i)_{i\in I}\right) = \mathcal{U} - \lim_{i\in I} d_S^{M_i}(a_i, b_i).$$

Define the interpretation of the sort S in $\prod_{i\in I}^{\mathcal{U}} M_i$ as the metric space $X_S^{\mathcal{U}}$ obtained from $\widetilde{X}_S^{\mathcal{U}}$ and the pseudometric $\widetilde{d}_S$. If $D \in \mathcal{D}_S$, define

$$\left\{ \left[(a_i)_{i\in I}\right] \,|\, \text{for } \mathcal{U}\text{-a.a. } i \in I,\, a_i \in M_i(D) \right\}$$

Being $M(D)$ the metric ultraproduct of the complete metric spaces $(M_i(D))_{i\in I}$, $M(D)$ is a complete metric space. If $f : S_1 \times .... \times S_n \to S$ is a sorted function symbol, the interpretation of f in M is the function from $M(S_1) \times ..... \times M(S_n)$ to $M(S)$, defined by

$$f^{M_i}\left(\left[(a_i^1)_{i\in I}\right], ..., \left[(a_i^n)_{i\in I}\right]\right) = \left[\left(f^{M_i}\left(a_i^1, ..., a_i^n\right)\right)_{i\in I}\right]$$

If R is a sorted relation symbol on $S_1 \times ... \times S_n$, the interpretation of R in M is the function from $M(S_1) \times ..... \times M(S_n)$ to $\mathbb{R}$ defined by

$$R^M = \mathcal{U} - \lim_{i\in I} R^{M_i}$$

Observe that, by the boundedness and uniform continuity requirement on R^{M_i} and f^{M_i} restricted to domains, R^M and f^{M_i} are well defined and satisfy the same boundedness and uniform continuity requirements.

If $M_i = M$ for every $i \in I$, the ultraproduct is called ultrapower of M and denoted by $M^{\mathcal{U}}$.

Teorema 6.4.1 (Łos) *If $(M_i)_{i\in I}$ is a family of $\mathcal{L}$-structures, $\mathcal{U}$ is a ultrafilter on I and $M = \prod_i^{\mathcal{U}} M_i$, then*

1. *for every $\mathcal{L}$-formula φ with free variables $x_1, ..., x_n$ of sorts $S_1, ..., S_n$,*

$$\varphi^M\left(\left[(a_i^1)_{i\in I}\right], ..., \left[(a_i^n)_{i\in I}\right]\right) = \mathcal{U} - \lim_{i\in I} \varphi^{M_i}\left(a_i^1, ..., a_i^n\right)$$

2. *every $\mathcal{L}$-sentence η,*

$$\eta^M = \mathcal{U} - \lim_{i\in I} \eta^{M_i}$$

Proof. The proof of point 1 is easily done by induction on the complexity of the formula. Points 2 follows. ■

6.5 Character density of languages

If $\mathcal{L}$ is a language, $\mathcal{T}$ is an $\mathcal{L}$-theory, φ, ψ are formulae with free variables $\overline{x}$ of sort $\overline{S}$ and $\overline{D} = (D_1, ..., D_n)$, where $D_i \in \mathcal{D}_{S_i}$ $\forall i \in \{1, 2, ..., n\}$, we set

$$d_{\overline{D}}^{\mathcal{T}}(\varphi, \psi) = \sup\left\{|\varphi(\overline{a}) - \psi(\overline{a})| \,\middle|\, \overline{a} \in M\left(\overline{D}\right), \, M \models \mathcal{T}\right\}$$

This defines a pseudo-metric on the set of such formulae, whose character density is denoted by $\chi_{\mathcal{L}}\left(\overline{D}, \mathcal{T}\right)$. The **character density** of the $\mathcal{L}$-theory $\mathcal{T}$ is

$$\chi_{\mathcal{L}}(\mathcal{T}) = \sum_{\overline{D}} \chi_{\mathcal{L}}\left(\overline{D}, \mathcal{T}\right).$$

If $\mathcal{T}$ is the empty theory, $d_{\overline{D}}^{\mathcal{T}}$ is denoted by $d_{\overline{D}}$, $\chi_{\mathcal{L}}\left(\overline{D}, \mathcal{T}\right)$ by $\chi_{\mathcal{L}}\left(\overline{D}\right)$ and $\chi_{\mathcal{L}}(\mathcal{T})$ by $\chi_{\mathcal{L}}$. We call $\chi_{\mathcal{L}}$ the character density of the language.

The character density of an $\mathcal{L}$-structure M is

$$\chi(M) = \sum_{S} \chi(M(S))$$

where S ranges over all sorts and $\chi(M(S))$ is the character density of the metric space $\left(M(S), d_S^M\right)$.

Lemma 6.5.1 (Tarski-Vaught criterion) *If $N \subset M$ are $\mathcal{L}$-structures and, for every choice of domains $\overline{D}$ of sorts $\overline{S}$, there is a set $\mathcal{F}_{\overline{D}}$ of$\mathcal{L}$-formulae which is dense in the set of formulae with parameters in N and free variables $\overline{x}$ of sorts $\overline{S}$ with respect to the metric $d_{\overline{D}}$ such that, for every $\varphi, \psi \in \mathcal{F}$*

$$\inf_{\overline{a} \in N\left(\overline{D}\right)} \varphi(\overline{a}) = \inf_{\overline{a} \in M\left(\overline{D}\right)} \varphi\left(\overline{b}\right),$$

then $N \prec M$

Proof. By density of $\mathcal{F}_{\overline{D}}$, the condition holds for every formula with parameters in N and free variables $\overline{x}$ of sorts $\overline{S}$. The fact that $\varphi^M = \varphi^N$ for every such formula follows now by induction on the complexity. ■

Teorema 6.5.2 (Downward Löwenheim-Skolem) *If M is an $\mathcal{L}$-structure and $X \subset M$, then there is $N \prec M$ such that $X \subset N$ and $\chi(N) \leq \chi_{\mathcal{L}}(Th(M)) + \chi(X)$*

Proof. For every choice of domains, fix a set $\mathcal{F}_{\overline{D}}$ of formulae which is dense in the set of formulae with parameters in N and free variables $\overline{x}$ of sorts $\overline{S}$ with respect to the metric $d_{\overline{D}}$, in such a way that

$$\left|\bigcup_{\overline{D}} \mathcal{F}_{\overline{D}}\right| \leq \chi_{\mathcal{L}}(Th(M))$$

Define recursively sets $(X_n)_{n\in\mathbb{N}}$ and $(E_n)_{n\in\mathbb{N}}$ such that, $\forall n\in\mathbb{N}$, E_n is dense in X_n, $\chi(X_n)\leq\chi_{\mathcal{L}}(Th(M))+\chi(X)$ and, for every positive rational number r, $k\in\mathbb{N}$, domain $\overline{D}$ and formula $\varphi\in\mathcal{F}_{\overline{D}}$ with parameters from $\bigcup_{j<n}E_n$, if

$$\inf_{\overline{x}\in M(\overline{D})}\varphi(\overline{x})\leq r$$

then there is $b\in X_n$ such that

$$\varphi\left(\overline{b}\right)\leq r+\frac{1}{k}$$

By the Tarski-Vaught criterion, the closure of $\bigcup_n X_n$ is an elementary submodel of M containing X with character density at most $\chi(X)+\chi_{\mathcal{L}}(Th(M))$. ■

Chapter 7

Stability

7.1 Types

If $\mathcal{L}$ is a language, "$\varphi(\overline{x}) \leq r$" is a condition and $\overline{D}$ a choice of domains compatible with $\overline{x}$, we say that it is satisfied in an $\mathcal{L}$-structure M by $\overline{a} \in M(\overline{D})$ if $\varphi^M(\overline{a}) \leq r$. A set $\Sigma(\overline{x})$ of such conditions is satisfied by $\overline{a} \in M$ if every element of $\Sigma(\overline{x})$ is satisfied by $\overline{a} \in M(\overline{D})$.

Proposizione 7.1.1 (Compactness) *If $\Sigma(\overline{x})$ is a set of conditions, TFAE*

1. *$\Sigma(\overline{x})$ is $\overline{D}$-satisfiable, i.e. there is an $\mathcal{L}$-structure M and $\overline{a} \in M(\overline{D})$ that satisfies $\Sigma(\overline{x})$*
2. *$\Sigma(\overline{x})$ is finitely $\overline{D}$-satisfiable, namely every finite subset of $\Sigma(\overline{x})$ is $\overline{D}$-satisfiable*
3. *$\Sigma(\overline{x})$ is finitely approximately $\overline{D}$-satisfiable, namely for every finite subset $\mathcal{F}$ of $\Sigma(\overline{x})$ and every $\varepsilon > 0$ there is an $\mathcal{L}$-structure M and $\overline{a} \in M(\overline{D})$ such that, for every condition "$\varphi \leq r$" in $\mathcal{F}$, $\overline{a}$ satisfies "$\varphi \leq r + \varepsilon$"*

A satisfiable set of conditions $\Sigma(\overline{x})$ is called a partial $\overline{D}$-type. If M is an $\mathcal{L}$-structure and $\overline{a} \in M(\overline{D})$ satisfies every condition in $\Sigma(\overline{x})$, then $\overline{a}$ is called a realization of $\Sigma(\overline{x})$ in M. A maximal partial $\overline{D}$-type is called a $\overline{D}$-type. If M is an $\mathcal{L}$ -structure and $\overline{a} \in M(\overline{D})$,

$$tp^M(\overline{a}) = \left\{ "\varphi \leq r" \,\middle|\, \varphi^M(\overline{a}) \leq r \right\}$$

is the type of $\overline{a}$ in M. It is easily seen that this is a type. Conversely, by compactness, every type has this form. More precisely, if p is a $\overline{D}$-type and $\overline{a}$ is a realization of p in M, then $p = tp^M(\overline{a})$.

Define $S_{\mathcal{L}}(\overline{D})$ the set of $\overline{D}$-types in the language $\mathcal{L}$. If φ is a formula, then $\varphi^M(\overline{a})$ does not depend on the particular realization $\overline{a}$ of p in M chosen. Therefore, it is well defined the real number $\varphi^p = \varphi^M(\overline{a})$, where $\overline{a}$ is a realization of p in M. Thus, any formula can be seen as a function from $S_{\mathcal{L}}\left(\overline{D}\right)$ to $\mathbb{R}$. Moreover, two formulae are equivalent iff the functions they define are the same.

If "$\varphi \leq r$" is a condition, denote by $[\varphi \leq r]$ the set of $\overline{D}$-types that contain the condition $\varphi \leq r$ or, equivalently, such that $\varphi^p \leq r$. Analogously, if Σ is any partial type, define $[\Sigma] = \left\{p \in S_{\mathcal{L}}\left(\overline{D}\right) \mid \Sigma \subset p\right\}$. The family of such sets is a family of closed sets for a compact Hausdorff topology on $S_{\mathcal{L}}\left(\overline{D}\right)$, called the logic topology, which is the weakest topology making all the formulae continuous. Moreover, a function turns out to be continuous iff it is a uniform limit of formulae.

On $S_{\mathcal{L}}\left(\overline{D}\right)$ one can define also a metric d, setting

$$d\left(p,q\right) = \inf\left\{d^M\left(\overline{a},\overline{b}\right) \,\middle|\, \overline{a},\overline{b} \in M \text{ satisfy } p,q \text{ respectively}\right\}$$

where the infimum is actually a minimum by compactness. The topology induced by d is finer than the logic topology. In fact, suppose $(p_i)_{i\in I}$ is a net in $[\varphi \leq r]$ converging to $p \in S_{\mathcal{L}}\left(\overline{D}\right)$ in this metric. If $\varepsilon > 0$, there is a $\delta > 0$ such that, for every $\mathcal{L}$-structure M, if $d^M\left(\overline{a},\overline{b}\right) < \delta$ then $\left|\varphi^M\left(\overline{a}\right) - \varphi^M\left(\overline{b}\right)\right| < \varepsilon$. Thus, if $i \in I$ is such that $d\left(p_i,p\right) < \delta$ then, if M is an $\mathcal{L}$-structure and $\overline{a}_i, \overline{a} \in M$ satisfy p_i, p in M and $d^M\left(\overline{a}_i,\overline{a}\right) < \delta$, then $\varepsilon > \left|\varphi^M\left(\overline{a}_i\right) - \varphi^M\left(\overline{a}\right)\right| = \left|\varphi^{p_i} - \varphi^p\right|$ and hence, since $\varphi^{p_i} \leq r$, $\varphi^p \leq r + \varepsilon$. Since this is true for every $\varepsilon > 0$, $\varphi^p \leq r$ and hence $p \in [\varphi \leq r]$. This shows that $[\varphi \leq r]$ is closed in the metric d. If F is a logically closed subset, then $F^{\varepsilon} = \bigcap_{n\in\mathbb{N}}\left\{q \in \mathbb{N} \,\middle|\, \exists p \in F,\, d\left(p,q\right) < \varepsilon + \frac{1}{n}\right\}$ is still logically closed. Thus, $S_{\mathcal{L}}\left(\overline{D}\right)$ is a so called compact called topometric space.

If T is a satisfiable $\mathcal{L}$-theory, then $S_{\mathcal{L}}\left(\overline{D},T\right)$ is the closed subspace $[T]$ of $S_{\mathcal{L}}\left(\overline{D}\right)$ of the types that can be realized in a model of T or, equivalently, that contain T. If M is an $\mathcal{L}$-structure and $A \subset M$, the set $S_{\mathcal{L}}\left(\overline{D},A\right)$ of types over A is by definition $S_{\mathcal{L}(A)}\left(\overline{D}, Th\left((M,a)_{a\in A}\right)\right)$.

7.2 Saturation

If κ is a cardinal, an $\mathcal{L}$-structure is **κ-saturated** if, for every $A \subset M$ of character density (or, equivalently, cardinality) $< \kappa$, every type over A is realized in M. As usual, $\aleph_1$-saturation is referred to as countable saturation. An $\mathcal{L}$-structure M is said saturated if it is $\chi\left(M\right)$-saturated, where $\chi\left(M\right)$ is the character density of M.

The classic Keisler theorem on ultraproduct holds without changes for the logic for metric structures.

Proposizione 7.2.1 *If $\mathcal{L}$ is a separable language, $(M_i)_{i\in I}$ is a family of $\mathcal{L}$-structures and $\mathcal{U}$ is a countably incomplete ultrafilter over I, then the ultraproduct $M = \prod_{i\in I}^{\mathcal{U}} M_i$ is countably saturated*

Proof. Suppose p is a $\overline{D}$-type over $A \subset M$, where A has cardinality $\leq \aleph_0$.

Observe that

$$\left(\prod_{i\in I}^{\mathcal{U}} M_i, \left[(a_i)_{i\in I}\right]_{\mathcal{U}}\right)_{\left[(a_i)_{i\in I}\right]_{\mathcal{U}}\in A} = \prod_{i\in I}^{\mathcal{U}} (M_i, a_i)_{\left[(a_i)_{i\in I}\right]_{\mathcal{U}}\in A}$$

Thus, replacing $\mathcal{L}$ with $\mathcal{L}(A)$, where $\left[(a_i)_{i\in I}\right]_{\mathcal{U}} \in A$ is interpreted as a_i in M_i, we can suppose $A = \varnothing$. Suppose $q = \{"\varphi_n \leq r_n" \mid n \in \mathbb{N}\}$ is a countable dense subset of p. In order to show that p is realized in M, it suffices to show that q is realized in M. Consider a sequence $\{I_n\}_{n\in\omega}$ of elements of I such that $I_0 = I$, $I_{n+1} \subset I_n \in \mathcal{U}$ for every $n \in \omega$ and $\bigcap_{n\in\mathbb{N}} I_n = \varnothing$. Define recursively $J_0 = I_0 = I$ and, for every $n \in \mathbb{N}$,

$$J_n = \left\{ i \in I_n \cap J_{n-1} \;\middle|\; M_i \models \inf_{\overline{b}\in\overline{D}} \max\left\{ \max_{1\leq j\leq n} \left(\varphi_j\left(\overline{b}\right) - \left(r_j + \frac{1}{n}\right)\right), 0\right\} \right\} \in \mathcal{U}$$

Define now, for every $i \in I$,

$$n(i) = \min\{n \in \omega \mid i \notin J_{n+1}\}.$$

If $i \in I$ and $n(i) \geq 1$, define $\overline{b}(i) \in M_i\left(\overline{D}\right)$ such that

$$M_i \models \max\left\{ \max_{1\leq j\leq n(i)} \left(\varphi_j\left(\overline{b}(i)\right) - \left(r_j + \frac{2}{n(i)}\right)\right), 0\right\}$$

Define $\overline{b} = \left[\left(\overline{b}_i\right)_{i\in I}\right] \in \prod_{i\in I}^{\mathcal{U}} M_i$. If $n \in \mathbb{N}$ and $i \in J_n$ then $n(i) \geq n$ and hence

$$M_i \models \max\left\{ \max_{1\leq j\leq n} \left(\varphi_j\left(\overline{b}(i)\right) - \left(r_j + \frac{2}{n}\right)\right), 0\right\}$$

Since $J_n \in \mathcal{U}$, this implies that

$$\prod_{i\in I}^{\mathcal{U}} M_I \models \max\left\{ \max_{1\leq j\leq n} \left(\varphi_j\left(\overline{b}\right) - \left(r_j + \frac{2}{n}\right)\right), 0\right\}$$

and, since this is true for every $n \in \mathbb{N}$,

$$\prod_{i\in I}^{\mathcal{U}} M_i \models \max\left\{ \max_{n\in\mathbb{N}} \left(\varphi_j\left(\overline{b}\right) - r_j\right), 0\right\}$$

■

Proposizione 7.2.2 *If $\mathcal{L}$ is a language and M, N are two elementarily equivalent saturated $\mathcal{L}$-structures of the same character density κ, then M and N are isomorphic*

Proof. Suppose $\{a_i\}_{i<\kappa}$ and $\{b_i\}_{i<\kappa}$ are dense subsets of M and N respectively. If $i < \kappa$ is an ordinal, write $i = j + n$ where j is a limit ordinal and $n \in \omega$. Say that i is even (resp. odd) if such is n. Now I define recursively sequences $\{\widetilde{a}_i\}_{i<\kappa}$

and $\left\{\widetilde{b}_i\right\}_{i<\kappa}$ such that, for every $i<k$, if $i=j+2n$ is even, then $\widetilde{a}_{j+2n}=a_{j+n}$, if $i=j+2n+1$ is odd, then $\widetilde{b}_{j+2n+1}=b_{j+n}$, and such that, for every $i<\kappa$, the structures $(M,\widetilde{a}_j)_{j<i}$ and $\left(N,\widetilde{b}_j\right)_{j<i}$ are elementarily equivalent. Suppose these sequences have been defined for $i<\lambda=\delta+n$, where δ is a limit ordinal and $n\in\omega$. Suppose without loss of generality that $n=2m$ is even. Define $\widetilde{a}_\lambda=a_{\delta+m}$ and consider the complete D-type p of $\widetilde{a}_\lambda\in M(D)$ over $\{\widetilde{a}_i\}_{i<\lambda}$. If q is the D-type over $\left\{\widetilde{b}_i\right\}_{i<\lambda}$ obtained by p replacing every $\widetilde{a}_i$ with $\widetilde{b}_i$, then q is a complete D-type and, by saturation of N, there is $\widetilde{b}_\lambda\in N(D)$ that satisfies q. Define then $\widetilde{b}_{\lambda+1}=b_{\delta+m}\in N(D')$ and find $\widetilde{a}_{\lambda+1}\in M(D')$ as before. This concludes that recursive construction. Now, since the structures $(M,\widetilde{a}_i)_{i<\kappa}$ and $\left(N,\widetilde{b}_j\right)_{j<\kappa}$ are elementarily equivalent, the function Φ sending $\widetilde{a}_i$ to $\widetilde{b}_i$ is an isometric isomorphism, that can be extended to an isometric isomorphism from M onto N. ■

Corollario 7.2.3 *If CH holds, ultrapowers of elementarily equivalent $\mathcal{L}$-structures of character density $\leq\mathfrak{c}$ are isomorphic*

7.3 Stability

If λ is a cardinal and $\mathcal{L}$ is a language, a theory $\mathcal{T}$ is said to be **λ-stable** if, for every model M of $\mathcal{T}$, every $A\subset M$ of density character (or, equivalently, cardinality) $\leq\lambda$ and choice of domains $\overline{D}$, the space $S_{\mathcal{L}}\left(\overline{D},A\right)$ of complete $\overline{D}$-types over $A\subset M$ has character density $\leq\lambda$ with respect to the metric topology. A theory is **stable** if it is λ-stable for some λ, **unstable** otherwise. Observe that, by Lowenheim-Skolem, if $\mathcal{L}$ is separable, $\mathcal{T}$ is λ-stable iff, for every model M of $\mathcal{T}$ of density character (or, equivalently, cardinality) $\leq\lambda$ and choice of domains $\overline{D}$, $S_{\mathcal{L}}\left(\overline{D},M\right)$ has character density $\leq\lambda$.

If $\psi(\overline{x},\overline{y})$ is an $\mathcal{L}$-formula, where $\overline{x}$ and $\overline{y}$ are of the same sort $\overline{S}$, $\overline{D}$ is a choice of domains compatible with $\overline{x}$, $\varepsilon>0$ and M is an $\mathcal{L}$-structure, define the following relation $\prec^{\overline{D}}_{\psi,\varepsilon}$ on $M\left(\overline{S}\right)$:

$$\overline{a}\prec^{\overline{D}}_{\psi,\varepsilon}\overline{b}$$

iff

$$\psi^M\left(\overline{a},\overline{b}\right)\in[0,\varepsilon)$$

and

$$\psi^M\left(\overline{b},\overline{a}\right)\in(1-\varepsilon,1].$$

Denote $\prec^{\overline{D}}_{\psi,0}$ by $\prec^{\overline{D}}_{\psi}$.

An $\mathcal{L}$-structure has the order property if there is a formula $\psi(\overline{x},\overline{y})$ and a compatible choice of domains $\overline{D}$ such that $M\left(\overline{D}\right)$ contains an infinite $\prec^{\overline{D}}_{\psi}$-chain.

A sequence $(M_n)_{n\in\mathbb{N}}$ of $\mathcal{L}$-structures has the order property if there is a formula $\psi(\overline{x},\overline{y})$ and a compatible choice of domains $\overline{D}$ such that, $\forall n\in\mathbb{N}$, $\exists N\in\mathbb{N}$ such that, $\forall m\geq n$, M_m contains a $\prec_{\psi}^{\overline{D}}$-chain of length m.

An $\mathcal{L}$-structure M has the approximate order property iff there is a formula $\psi(\overline{x},\overline{y})$ and a compatible choice of domains $\overline{D}$ such that $M\left(\overline{D}\right)$ contains arbitrarily long finite $\prec_{\psi}^{\overline{D}}$-chains.

Observe that, for every $n\in\mathbb{N}$, "$M\left(\overline{D}\right)$ contains a $\prec_{\psi}^{\overline{D}}$-chain of length n" can be expressed by an $\mathcal{L}$-formula, and hence it is satisfied in every $\mathcal{L}$-structure N elementarily equivalent to M.

Finally, we say that a $\overline{D}$-type p over M is **finitely determined** if for every formula $\varphi(\overline{x},\overline{y})$, where the sort of $\overline{x}$ is consistent with $\overline{D}$, $\varepsilon>0$ and choice of domains $\overline{D}'$ consistent with $\overline{y}$, there is $\delta>0$ and a finite subset $\overline{B}$ of $M\left(\overline{D}\right)$ such that, for every $\overline{c}_1,\overline{c}_2\in M\left(\overline{D}'\right)$,

$$\sup_{\overline{b}\in\overline{B}}\left|\varphi\left(\overline{b},\overline{c}_1\right)-\varphi\left(\overline{b},\overline{c}_2\right)\right|\leq\delta$$

implies

$$\left|\varphi^p\left(\overline{x},\overline{c}_1\right)-\varphi^p\left(\overline{x},\overline{c}_2\right)\right|<\varepsilon$$

or, equivalently, that the condition

$$\left|\varphi\left(\overline{x},\overline{c}_1\right)-\varphi\left(\overline{x},\overline{c}_2\right)\right|<\varepsilon$$

belongs to p.

Lemma 7.3.1 *If M is a model of $\mathcal{T}$ which has a non-finitely determined type over it, then M has the order property*

Proof. By hypothesis, there is $p\in S_{\mathcal{L}}\left(\overline{D},M\right)$ and a formula $\psi(\overline{x},\overline{y})$, where the sort of $\overline{x}$ is compatible with $\overline{D}$, a choice of domains $\overline{D}'$ compatible with the sort of $\overline{y}$ and $\varepsilon\in\left(0,\frac{1}{2}\right)$ such that, for every $\delta>0$ and every finite subset B of $M\left(\overline{D}\right)$, there are $\overline{b}(\delta,B),\overline{c}\left(\delta,\overline{B}\right)\in M\left(\overline{D}'\right)$ such that

$$\sup_{\overline{a}\in\overline{B}}\left|\varphi\left(\overline{a},\overline{b}(\delta,B)\right)-\varphi\left(\overline{a},\overline{c}(\delta,B)\right)\right|\leq\delta$$

and

$$\left|\varphi^p\left(\overline{x},\overline{b}(\delta,B)\right)-\varphi^p\left(\overline{x},\overline{c}(\delta,B)\right)\right|\geq\varepsilon$$

Define now recursively sequences $(\alpha_n)_{n\in\omega}$ in $M\left(\overline{D}\right)$,$(\beta_n)_{n\in\omega}$, $(\gamma_n)_{n\in\omega}$ in $M\left(\overline{D}'\right)$ and $(B_n)_{n\in\omega}$ in $\left[M\left(\overline{D}\right)\right]^{<\aleph_0}$ in this way: $B_0=\varnothing$, $\beta_j=\overline{b}\left(\frac{\varepsilon}{2},B_j\right)$, $\gamma_j=\overline{c}\left(\frac{\varepsilon}{2},B_j\right)$, α_j realizing the finite subset

$$\left\{\text{"}\left|\varphi\left(\overline{x},\beta_i\right)-\varphi\left(\overline{x},\gamma_i\right)\right|\geq\varepsilon\text{"}\mid i\in\{0,1,...,j\}\right\}$$

of p and $B_{j+1} = B_j \cup \{\alpha_j, \beta_j, \gamma_j\}$. If f is a continuous function that is constantly equal to 0 on $(-\infty, \frac{\varepsilon}{2}]$ and constantly equal to 1 on $[\varepsilon, +\infty)$, then the formula

$$\theta(x_1, y_1, z_1, x_2, y_2, z_2) = f(|\varphi(x_1, y_2) - \varphi(x_1, z_2)|)$$

orders the sequence

$$((\alpha_n, \beta_n, \gamma_n))_{n\in\mathbb{N}}$$

in $M\left(\left(\overline{D}, \overline{D}', \overline{D}'\right)\right)$. ■

Proposizione 7.3.2 *If $\mathcal{L}$ is a separable language and $\mathcal{T}$ is an $\mathcal{L}$-theory, the following statements are equivalent*

1. *$\mathcal{T}$ is unstable*
2. *$\mathcal{T}$ is not $\mathfrak{c}$-stable*
3. *there is a model of $\mathcal{T}$ with the order property*
4. *there is a separable model of $\mathcal{T}$ with the order property*
5. *for every linear order I there is a formula $\psi(\overline{x}, \overline{y})$, a choice of domains $\overline{D}$ and a model M of $\mathcal{T}$ such that $M(\overline{D})$ contains a $\prec_\psi^{\overline{D}}$-chain of order type I*
6. *there is a model of $\mathcal{T}$ with the approximate order property*
7. *every model of $\mathcal{T}$ of density character $\mathfrak{c}$ has the order property*
8. *some model of $\mathcal{T}$ has non-finitely defined types over it*
9. *every model of $\mathcal{T}$ of density character $\mathfrak{c}$ has not finitely defined types over it*

Proof.

$1 \Rightarrow 2$ Obvious

$3 \Leftrightarrow 4 \Leftrightarrow 5 \Leftrightarrow 6$ By compactness, Lowenheim-Skolem and the Fundamental Theorem on Ultrafilters.

$5 \Rightarrow 1$ Fix a cardinal λ and suppose μ is the least ordinal (or cardinal) such that $2^\mu > \lambda$. Suppose $\psi(\overline{x}, \overline{y})$ is a formula, $\overline{D}$ is a choice of domains witnessing the order property, and M is a model of $\mathcal{T}$ with $\prec_\psi^{\overline{D}}$-chain $(\overline{a}_i)_{i\in 2^{<\mu}}$ of order type $2^{<\mu}$, where $2^{<\mu}$ has the lexicographic order. If

$$A = \{\overline{a}_i\}_{i\in 2^{<\mu}} \subset M(\overline{D})$$

then A is closed and discrete of cardinality $\leq \lambda$. By Lowenheim-Skolem, one can assume M has character density $\leq \lambda$. I now claim that the space $S_{\mathcal{L}}(\overline{D}, M)$ has metric character density $2^\mu > \lambda$. In fact, identify $2^{<\mu}$ with the subset of eventually zero sequences in 2^μ. For every $\sigma \in 2^\mu$ consider a $p_\sigma \in S_{\mathcal{L}}\left(\overline{D}, M\right)$ containing the consistent type

$$\left\{\varphi\left(\overline{x}, \overline{a}_\tau\right) = 0 \,\middle|\, \tau > \sigma,\, \tau \in 2^{<\mu}\right\} \cup \left\{\varphi\left(\overline{x}, \overline{a}_\tau\right) = 1 \,\middle|\, \tau < \sigma, \tau \in 2^{<\mu}\right\}$$

I claim that $E = \{p_\sigma : \sigma \in 2^\mu\}$ is a closed and discrete subset of $S_{\mathcal{L}}\left(\overline{D}, M\right)$ in the metric d of cardinality $2^\mu > \lambda$. Suppose $\varepsilon > 0$ is such that, for every $N \succ M$ and $\overline{b}, \overline{d}, \overline{b}', \overline{d}' \in N\left(\overline{D}\right)$, $\max\left\{d\left(\overline{b}, \overline{b}'\right), d\left(\overline{d}, \overline{d}'\right)\right\} < \varepsilon$ implies $\left|\psi\left(\overline{b}, \overline{d}\right) - \psi\left(\overline{b}', \overline{d}'\right)\right| < 1$. Consider $\sigma \neq \sigma' \in 2^\mu$ and $\alpha \in \mu$ such that $\sigma_{|\alpha} = \sigma'_{|\alpha}$, $\sigma(\alpha) = 0$ and $\sigma'(\alpha) = 1$. Suppose $N \succ M$ and $\overline{c}, \overline{c}'$ realize σ, σ'. Thus, $\psi\left(\overline{c}, a_{\sigma_{|\alpha+1}}\right) = 1$ and $\psi\left(\overline{c}', a_{\sigma_{|\alpha+1}}\right) = 0$. As a consequence, $\left|\psi\left(\overline{c}', a_{\sigma_{|\alpha+1}}\right) - \psi\left(\overline{c}, a_{\sigma_{|\alpha+1}}\right)\right| \geq 1$ and $d\left(\overline{c}, \overline{c}'\right) \geq \varepsilon$. This implies that $d\left(p_\sigma, p_{\sigma'}\right) \geq \varepsilon$. This shows that the metric character density of $S_{\mathcal{L}}\left(\overline{D}, M\right)$ is at least $|E| = 2^\mu > \lambda$. This shows that $\mathcal{T}$ is λ-unstable.

$2 \Rightarrow 9$ Suppose M is a model of $\mathcal{T}$ of character density $\mathfrak{c}$. If, by contradiction, every type over M is finitely determined, then the character density of $S_{\mathcal{L}}\left(\overline{D}, M\right)$ is at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$

$9 \Rightarrow 8 \wedge 7 \Rightarrow 3$ Obvious

$9 \Rightarrow 7 \wedge 8 \Rightarrow 3$ It follows from the previous lemma

■

Corollario 7.3.3 *If $\mathcal{L}$ is a separable language and A is an $\mathcal{L}$-structure, then the complete theory $Th(A)$ of A is unstable iff A has the approximate order property*

Proof. Sufficiency is obvious. About necessity, assuming $Th(A)$ unstable, by $1 \Rightarrow 6$ of the previous proposition there is a model M of $Th(A)$ that has the approximate order property. Thus, M is elementarily equivalent to A and, since the approximate order property can be expressed by $\mathcal{L}$-formulae, A has the approximate order property as well. ■

7.4 Gaps and the order property

If $(P, \leq)$ is a poset and λ, μ two ordinals, a (λ, μ)**-pregap** in P is an increasing sequence $(a_i)_{i\in\lambda}$ indexed by λ of elements of P and a decreasing sequence $(b_j)_{j\in\mu}$ in P such that $a_i \leq b_j$ for every $i \in \lambda$ and $j \in \mu$. An element x of P such that $a_i \leq x \leq b_j$ fills or separates the pregap. A (λ, μ)-**gap** is a (λ, μ)-pregap which is not filled.

If L is a linear order, the coinitiality of L is the minimal cardinality of a subset X of L such that, for every $x \in L$ there is $y \in X$ such that $y \leq x$. Denote by $\mathbb{N}^{\nearrow\mathbb{N}}$ the set of nondecreasing functions f from $\mathbb{N}$ to $\mathbb{N}$ such that $\lim_{n\to+\infty} f(n) = +\infty$. If $\mathcal{U}$ is a ultrafilter over $\mathbb{N}$, the set $\mathbb{N}^{\nearrow\mathbb{N}}/\mathcal{U}$ of equivalence classes of elements of $\mathbb{N}^{\nearrow\mathbb{N}}$ modulo $\mathcal{U}$ is linearly ordered by the relation $[f]_{\mathcal{U}} \leq [g]_{\mathcal{U}}$ iff $\{n \in \mathbb{N} \mid f(n) \leq g(n)\} \in \mathcal{U}$. The coinitiality of $\mathbb{N}^{\nearrow\mathbb{N}}/\mathcal{U}$ is denoted by $\kappa(\mathcal{U})$.

Lemma 7.4.1 *If $(M_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{L}$-structures with the order property, $\psi(\overline{x}, \overline{y})$ is a formula witnessing the order property of such sequence, where $\overline{x}$ and $\overline{y}$ are of sort $\overline{S}$, $\overline{D}$ is a compatible choice of domains, and $\mathcal{U}$ is a ultrafilter on $\mathbb{N}$, then the least ordinal λ such that $\prod_n^{\mathcal{U}} M_n(\overline{D})$ contains an (ω, λ)-gap with respect to $\prec_\psi^{\overline{D}}$ is $\kappa(\mathcal{U})$.*

Proof. Suppose $\{[f_i]_{\mathcal{U}} : i \in \kappa(\mathcal{U})\}$ is a decreasing sequence in $\mathbb{N}^{\nearrow\mathbb{N}}/\mathcal{U}$ such that, for every $[g] \in \mathbb{N}^{\nearrow\mathbb{N}}/\mathcal{U}$, there is $i \in \kappa(\mathcal{U})$ such that $[f_i] \leq [g]$. Define recursively $N_0 = 1$ and, for every $m \in \mathbb{N}$, $N(m) \geq \max\{m, N_{m-1}\}$ such that, for every $i \geq N(m)$ there is a $\prec_{\psi,\frac{1}{m}}^{\overline{D}}$ chain $\left(\overline{a}_1^{i,m}, ..., \overline{a}_m^{i,m}\right)$ of length m in $M_i(\overline{D})$. For every $i \in I$, define $m(i) \in \mathbb{N}$ such that $i \in [N(m(i)), N(m(i+1))]$ and $\overline{a}_j^{i,m(i)} = \overline{a}_j^i$ for every $j \in \{1, 2, ..., m(i)\}$. If $h \in \mathbb{N}^{\mathbb{N}}$, define $\overline{\mathbf{a}}_h \in \prod_n^{\mathcal{U}} M_n$ by $\overline{\mathbf{a}}_h(i) = a_{h(i)}^i$ if $h(i) \leq m(i)$ and $\overline{\mathbf{a}}_h(i) = a_{m(i)}^i$ otherwise. For every $m \in \mathbb{N}$, denote by h_m the function from $\mathbb{N}$ to $\mathbb{N}$ constantly equal to m. I claim that $(\overline{\mathbf{a}}_{h_n})_{n\in\mathbb{N}}$ and $(\overline{\mathbf{a}}_{f_i})_{i\in\kappa(\mathcal{U})}$ form an (ω, κ)-gap in $M(\overline{D})$ with respect to $\prec_\psi^{\overline{D}}$. In fact, suppose $\overline{\mathbf{b}} \in \prod_n^{\mathcal{U}} M_n$ is such that $\overline{\mathbf{a}}_{h_n} \prec \overline{\mathbf{b}}$ for every $n \in \mathbb{N}$. Define, for every $m \in \mathbb{N}$,

$$X_m = \left\{ i \geq N(m) \,\middle|\, \forall k \in \{1, 2, ..., m\}, \overline{\mathbf{a}}_{h_k}(i) \prec_{\psi,\frac{1}{m}}^{\overline{D}} \overline{\mathbf{b}}(i) \right\}$$

Thus, $(X_m)_{m\in\mathbb{N}}$ is a decreasing sequence of elements of $\mathcal{U}$ such that $\bigcap_m X_m = \varnothing$. For $i \in X_1$, define $h(i) = m$ if $i \in X_m \backslash X_{m+1}$. Consider an element X of $\mathcal{U}$ such that, for every $m \in \mathbb{N}$, $X \backslash X_m$ is finite. Define, recursively, $K_0 = 1$ and, $\forall n \in \mathbb{N}$, $K_n \geq K_{n-1}$ such that $X \cap [K_n, +\infty) \subset X_n$. Define, for every $n \in \mathbb{N}$ and $x \in [K_n, K_{n+1})$, $h(x) = n$. This defines $\mathcal{U}$-almost everywhere an element h of $\mathbb{N}^{\nearrow\mathbb{N}}$ such that $\overline{\mathbf{a}}_h \prec_\psi^{\overline{D}} \overline{\mathbf{b}}$. If $i \in \kappa(\mathcal{U})$ is such that $[f_i]_{\mathcal{U}} \leq [h]_{\mathcal{U}}$, then $\overline{\mathbf{a}}_{f_i} \prec_\psi^{\overline{D}} \overline{\mathbf{a}}_h \prec_\psi^{\overline{D}} \overline{\mathbf{b}}$. This shows that $\prod_n^{\mathcal{U}} M_n(\overline{D})$ contain an $(\omega, \kappa(\mathcal{U}))$-pregap. Suppose now that λ is an ordinal such that $\prod_n^{\mathcal{U}} M_n$ contains an (ω, λ)-gap and suppose $(\overline{\mathbf{a}}_n)_{n\in\mathbb{N}}$ and $\left(\overline{\mathbf{b}}_i\right)_{i<\lambda}$ give an (ω, λ)-gap on $\prod_n^{\mathcal{U}} M_n$. Define, for every $n \in \mathbb{N}$,

$$Y_n = \left\{ i \geq n \,\middle|\, \overline{\mathbf{a}}_1(i), ..., \overline{\mathbf{a}}_n(i) \text{ form a } \prec_{\psi,\frac{1}{n}}^{\overline{D}} \text{-chain} \right\} \in \mathcal{U}$$

Define, for every $m \in \mathbb{N}$ and $i \in Y_m \backslash Y_{m+1}$, $m(i) = m$. If $h \in \mathbb{N}^{\mathbb{N}}$, define, as above, $\overline{\mathbf{a}}_h \in \prod_n^{\mathcal{U}} M_n$ by, for $i \in Y_1$,

$$\overline{\mathbf{a}}_h(i) = \overline{\mathbf{a}}_{\min\{h(i), m(i)\}}(i)$$

and observe that $\overline{\mathbf{a}}_n \prec_\psi^{\overline{D}} \overline{\mathbf{a}}_h$ for every $n \in \mathbb{N}$ iff $\mathcal{U} - \lim_{n\in\mathbb{N}} h(n) = +\infty$. Reasoning as above, for every $i < \lambda$, it is possible to find $f_i \in \mathbb{N}^{\nearrow\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $\overline{\mathbf{a}}_n \prec_\psi^{\overline{D}} \overline{\mathbf{a}}_{f_i} \prec \overline{\mathbf{b}}_i$. Now, I claim that the family $\{[f_i]_{\mathcal{U}}\}_{i\in\lambda}$ is such that, for every $g \in \mathbb{N}^{\nearrow\mathbb{N}}$, there is $i \in \lambda$ such that $[f_i]_{\mathcal{U}} \leq [g]_{\mathcal{U}}$. In fact, suppose $g \in \mathbb{N}^{\nearrow\mathbb{N}}$. Thus, $\overline{\mathbf{a}}_n \prec \overline{\mathbf{a}}_g$ for every $n \in \mathbb{N}$ and, since $\overline{\mathbf{a}}_g$ does not separate the gap, there is $i \in \lambda$ such that $\overline{\mathbf{a}}_g \not\prec_{\psi}^{\overline{D}} \overline{\mathbf{b}}_i$. It follows that $\overline{\mathbf{a}}_g \not\prec_{\psi}^{\overline{D}} \overline{\mathbf{a}}_{f_i}$ and, hence, $[g]_{\mathcal{U}} \nleq [f_i]_{\mathcal{U}}$ and $[f]_{\mathcal{U}} \leq [g_i]_{\mathcal{U}}$. ■

In [D] it is proved that, for every regular cardinal μ such that $\aleph_0 \leq \mu \leq 2^{\aleph_0}$, there is a ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that $\kappa(\mathcal{U}) = \mu$. From this and the previous lemma follows directly the following theorems.

Teorema 7.4.2 *If $\mathcal{L}$ is a separable language, $(M_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{L}$-structures with the order property and $\neg CH$ holds, then there are nonprincipal ultrafilters $\mathcal{U}$,$\mathcal{V}$ over $\mathbb{N}$ such that*

$$\prod_n{}^{\mathcal{U}} M_n \ncong \prod_n{}^{\mathcal{V}} M_n$$

Proof. Pick ultrafilters $\mathcal{U}, \mathcal{V}$ over $\mathbb{N}$ such that $\kappa(\mathcal{U}) = \aleph_1$ and $\kappa(\mathcal{V}) = \aleph_2$. By the previous lemma, if ψ and $\overline{D}$ are a formula and a choice of domains witnessing the order property of the sequence $(M_n)_{n\in\mathbb{N}}$, then $\prod_n{}^{\mathcal{V}} M_n$ contains a (ω, ω_2)-gap with respect to $\prec_{\psi}^{\overline{D}}$, while $\prod_n{}^{\mathcal{U}} M_n$ does not. This implies that $\prod_n{}^{\mathcal{U}} M_n$ and $\prod_n{}^{\mathcal{V}} M_n$ are not isomorphic. ■

In [FS] this theorem is refined, getting the following

Teorema 7.4.3 *If $\mathcal{L}$ is a separable language, $(M_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{L}$-structures with the order property and $\neg CH$ holds, then there are $2^{2^{\aleph_0}}$ many nonisomorphic ultraproduct of this sequence*

Teorema 7.4.4 *If $\mathcal{L}$ is a separable language and A is an $\mathcal{L}$-structure of character density $\leq \mathfrak{c}$ whose complete theory is unstable, then the following statements are equivalent*

1. *for every nonprincipal ultrafilters $\mathcal{U}, \mathcal{V}$ over $\mathbb{N}$, $A^{\mathcal{U}} \simeq A^{\mathcal{V}}$*
2. *the Continuum Hypothesis holds*

Moreover, if $\neg CH$ holds, then there are $2^{\mathfrak{c}}$-many nonisomorphic ultrapowers of A

Proof.

$2 \Rightarrow 1$ If CH holds, then $A^{\mathcal{U}}$ is saturated for every nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$ and the result follows from the fact that any two saturated

$1 \Rightarrow 2$ Since $Th(A)$ is unstable, A has the approximate order property and the sequence constantly equal to A has the order property. The previous theorem can be applied.

■

7.5 Order property for C*-algebras

Lemma 7.5.1 *If M is an infinite-dimensional C*-algebra, then M has the order property*

Proof. Since any infinite-dimensional C*-algebra M has infinite-dimensional abelian *-subalgebras, we can assume M abelian. By the characterization theorem of abelian C*-algebras, there is a locally compact Hausdorff space X such that $M = C_0(X)$, If $\widehat{X} = X \cup \{\infty\}$ is the one point compactification of X, then

$$\begin{aligned} & C_0(X) \\ = & \left\{ f : X \to \mathbb{C} \,\middle|\, f \text{ is continuous and, } \forall \varepsilon > 0,\ (|f| - \varepsilon)^+ \text{ is compactly supported} \right\} \\ = & \left\{ g_{|X} \,\middle|\, g \in C\left(\widehat{X}\right),\ g(\infty) = 0 \right\}. \end{aligned}$$

Since M is infinite-dimensional, X is infinite. Consider an injective sequence $(a_n)_{n\in\mathbb{N}}$ of elements of X converging to ∞. Define recursively compact neighborhoods K_n of $\{a_1, ..., a_n\}$ and functions $f_n \in C_0(X)$ such that $f(X) \subset [0,1]$, $K_n \subset K_{n+1}$ and $f_n(K_n) = 1$ and $f_n(a_m) = 0$ for $m > n$. Suppose $n \geq 0$ and $f_1, ..., f_n$ and $K_1, ..., K_n$ have been defined. Define now K_{n+1} a compact neighborhood of $K_n \cup \{a_{n+1}\}$ missing $\{a_i\}_{i>n+1}$ and f_{n+1} a continuous function from X to $[0,1]$ such that $f_{n+1}(K_{n+1}) = 1$ and $f_{n+1}(a_m) = 0$ for $m > n+1$. Define now

$$g_n = \sup_{1 \leq k \leq n} f_k$$

and observe that $\{g_n\}_{n\in\mathbb{N}}$ is a sequence of distincts elements of M such that $g_n g_m = g_m$ if $m \leq n$ and $\|g_n - g_m\| = 1$ if $n \neq m$. Thus, if $\psi(x,y) = \|xy - y\|$ then $(g_n)_{n\in\mathbb{N}}$ is a $\prec_\psi^{D_1}$-chain and M has the order property. ■

Teorema 7.5.2 *If $\neg CH$ holds and M is an infinite-dimensional C*-algebra, then there are $2^{\mathfrak{c}}$-many nonisomorphic ultrapowers of M*

Teorema 7.5.3 *If M is a C*-algebra of character density $\leq \mathfrak{c}$ and CH holds, the for any two nonprincipal ultrafilters $\mathcal{U}$,$\mathcal{V}$ on $\mathbb{N}$, $M^{\mathcal{U}} \simeq M^{\mathcal{V}}$*

7.6 Order property for II_1 factors

Lemma 7.6.1 *There is a formula ψ in the language of von Neumann algebras such that, for every $n \in \mathbb{N}$, $\mathbb{M}_{2^n}$ contains a $\prec_\psi^{\overline{D}}$-chain of length $n-1$, where $\overline{D} = (D_1, D_1)$ and D_1 is the unit ball*

Proof. Identify $\mathbb{M}_{2^n}$ with $\mathbb{M}_2^{\otimes n}$. Consider

$$x = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$$

and observe that

$$[x, x^*] = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

and $\|x\|_2 = 1$, $\|[x, x^*]\|_2 = 2$. Define now, for $1 \leq i \leq n-1$,

$$a_i^n = \frac{1}{2} \bigotimes_{j=1}^{i} x \otimes \bigotimes_{j=i+1}^{n} 1$$

and

$$b_i^n = \frac{1}{2} \bigotimes_{j=1}^{i} 1 \otimes x^* \otimes \bigotimes_{j=i+2}^{j} 1.$$

Thus, $\|a_i^n\|_2 = \|b_i^n\|_2 = \frac{1}{2}$ and if $i \leq j$ then $\|[a_i, b_j]\|_2 = 0$, while $\|[a_i, b_j]\|_2 = 1$ if $j < i$. Thus, if $\psi(x_1, y_1, x_2, y_2) = \|[x_1, y_2]\|_2$, then $((a_i^n, b_i^n))_{i=1}^{n-1}$ is a $\prec_\psi^{\overline{D}}$-chain. ■

Corollario 7.6.2 *If M is a II_1 factor, then M has the approximate order property*

Proof. It follows from the fact that, for every $n \in \mathbb{N}$ there is an injective *-homomorphism $\Phi_n : \mathbb{M}_n \to M$ commuting with the trace. ■

Lemma 7.6.3 *The sequence $(\mathbb{M}_n)_{n\in\mathbb{N}}$ has the order property*

Proof. Suppose $\varepsilon > 0$ and $n \in \mathbb{N}$. If $m > \frac{2^n - 1}{\varepsilon}$, then $m = k \cdot 2^n + r$ for some $r \in \{0, 1, ..., 2^{n-1}\}$. Thus, if $p \in \mathbb{M}_m$ is such that $\tau(p) = \frac{k \cdot 2^n}{m}$, then $p\mathbb{M}_m p \simeq \mathbb{M}_{k \cdot 2^n} \simeq \mathbb{M}_k \otimes \mathbb{M}_{2^n}$. Identify $p\mathbb{M}_m p$ with $\mathbb{M}_k \otimes \mathbb{M}_{2^n}$. Suppose $((a_i^n, b_i^n))_{i=1}^n$ is as in the previous lemma. Define now $\alpha_i^n = 1 \otimes a_i^n \in \mathbb{M}_k \otimes \mathbb{M}_{2^n}$ and $\beta_i^n = 1 \otimes b_i^n \in \mathbb{M}_k \otimes \mathbb{M}_{2^n}$ and observe that, if $\psi(x_1, y_1, x_2, y_2) = \|[x_1, y_2]\|_2$ as before, then $((\alpha_i^n, \beta_i^n))_{i=1}^n$ is a $\prec_\psi^{(D_1, D_1)}$-chain in $\mathbb{M}_k \otimes \mathbb{M}_{2^n}$. Regard now α_i^n and β_i^n as elements of $\mathbb{M}_m$ and observe that, for every $x \in p\mathbb{M}_m p$,

$$\tau_{\mathbb{M}_m}(x) = \frac{k 2^n}{m} \tau_{p\mathbb{M}_m p}(x)$$

and hence

$$\|x\|_2^{p\mathbb{M}_m p} - \|x\|_2^{\mathbb{M}_m} = \frac{r}{m} \|x\|_2^{p\mathbb{M}_m p} < \varepsilon \|x\|_2^{p\mathbb{M}_m p}$$

for every $x \in \mathbb{M}_k \otimes \mathbb{M}_{2^n}$. Thus, $((\alpha_i^n, \beta_i^n))$ is a $\prec_{\psi,\varepsilon}^{(D_1, D_1)}$-chain in $\mathbb{M}_m$. ■

Teorema 7.6.4 *If $\neg CH$ holds, then there are $2^{\mathfrak{c}}$-many nonisomorphic ultraproducts of the sequence $(\mathbb{M}_n)_{n\in\mathbb{N}}$*

Teorema 7.6.5 *If $\neg CH$ holds and M is a II_1 factor, then there are $2^{\mathfrak{c}}$-many nonisomorphic ultrapowers of M*

Teorema 7.6.6 *If M is a von Neumann algebra of character density $\leq \mathfrak{c}$ and CH holds, then for any two nonprincipal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, $M^{\mathcal{U}} \simeq M^{\mathcal{V}}$*

7.7 Order property for symmetric and unitary groups

Lemma 7.7.1 *There is a formula $\psi(x_1, x_2, y_1, y_2)$ in the language of bounded bi-invariant metric groups such that, for every $n \in \mathbb{N}$, S_{3^n} has a $\prec_\psi$-chain of length n.*

Proof. Identify S_{3^n} with the set of permutations of $\{0,1,2\}^n$. Consider the inclusion

$$\overbrace{S_3 \times ... \times S_3}^{n \text{ times}} \hookrightarrow S_{\{0,1,2\}^n}$$

defined by

$$(\sigma_0, ..., \sigma_{n-1}) \to \sigma_0 \times ... \times \sigma_{n-1}$$

and

$$(\sigma_0 \times ... \times \sigma_{n-1})(i_0, ..., i_{n-1}) = (\sigma_1(i_1), ..., \sigma_n(i_n))$$

Define also, for $i = 1, ..., n$,

$$\sigma_i^n = \overbrace{(12) \times ... \times (12)}^{i \text{ times}} \times \overbrace{e \times ... \times e}^{n-i \text{ times}}$$

and

$$\tau_j^n = \overbrace{e \times ... \times e}^{j-1 \text{ times}} \times (23) \times \overbrace{e \times ... \times e}^{n-j \text{ times}}$$

Observe that, for $i < j$, $[\sigma_i, \tau_j] = 1$, and for $i \geq j$,

$$\left[\sigma_i^n, \tau_j^n\right] = \overbrace{e \times ... \times e}^{j-1 \text{ times}} \times (123) \times \overbrace{e \times ... \times e}^{i-j \text{ times}} \times \overbrace{e \times ... \times e}^{n-j \text{ times}}$$

Thus, if $i < j$, then

$$d\left(\left[\sigma_i^n, \tau_j^n\right], e\right) = 0$$

while, if $i \geq j$, then

$$d\left(\left[\sigma_i^n, \tau_j^n\right], e\right) = 1$$

Thus, if $\psi(x_1, x_2, y_1, y_2) = d([x_1, y_2], e)$, then the sequence $((\sigma_i, \tau_i))_{i=1}^n$ is a $\prec_\psi$-chain in S_{3^n}. ■

Lemma 7.7.2 *The sequence $(S_n)_{n\in\mathbb{N}}$ has the order property*

Proof. Suppose $\varepsilon > 0$ and $n \in \mathbb{N}$. Consider $m \in \mathbb{N}$ is such that $m > \frac{3^n}{\varepsilon}$ then $m = k3^n + r$ for some $k, r \in \mathbb{N}$ and $0 \leq r < 3^n$. Suppose that ψ and $((\sigma_i, \tau_i))_{i=1}^n$ are as in the previous lemma. As before, embed $\overbrace{S_{3^n} \times ... \times S_{3^n}}^{k \text{ times}}$ in S_{k3^n} and consider the elements

$$\Sigma_i^n = \overbrace{\sigma_i^n \times ... \times \sigma_i^n}^{k \text{ times}} \in S_{k3^n}$$

and

$$T_j^n = \overbrace{\tau_j^n \times ... \times \tau_j^n}^{k \text{ times}} \in S_{k3^n}.$$

Observe that $\left[\Sigma_i^n, T_j^n\right] = e$ if $i < j$ and $d\left(\left[\Sigma_i^n, T_j^n\right], e\right) = 1$ if $i \geq j$. Let S_{k3^n} act on the first $k3^n$ elements of m. This defines an inclusion of S_{k3^n} of S_m that sends Σ_i^n and T_j^n to elements $\widetilde{\Sigma}_i^{n,m}$ and $\widetilde{T}_j^{n,m}$ of S_m. These satisfy the following: $\left[\widetilde{\Sigma}_i^{n,m}, \widetilde{T}_j^{n,m}\right] = e$ if $i < j$ and

$$d\left(\left[\widetilde{\Sigma}_i^n, \widetilde{T}_j^n\right], e\right) = 1 - \frac{r}{m} > 1 - \frac{3^m}{m} > 1 - \varepsilon$$

if $i \geq j$. This shows that Since this true for every $\varepsilon > 0$, $n \in \mathbb{N}$ and $m > \frac{3^n}{\varepsilon}$, $(S_n)_{n\in\mathbb{N}}$ has the order property. ■

Corollario 7.7.3 *The sequence $(U_n)_{n\in\mathbb{N}}$ has the order property*

Proof. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Suppose $\eta > 0$ is such that $\sqrt{1-\eta} > 1 - \varepsilon$. If $m > \frac{3^n}{\eta}$, consider $\left(\left(\widetilde{\Sigma}_i^{n,m}, \widetilde{T}_i^{n,m}\right)\right)_{i=1}^n$ as in the proof of the previous lemma. Remind that, if $\sigma \in S_m$ and $A_\sigma \in U_m$ is defined by

$$A_\sigma(e_i) = e_{\sigma(i)}$$

then the function $\sigma \to A_\sigma$ is a homomorphism such that, for every $\sigma, \tau \in S_m$,

$$d(\sigma, \tau) = \frac{1}{2} d(A_\sigma, A_\tau)^2$$

Thus, if ψ is the formula as in the previous lemma, one has

$$\begin{aligned} \frac{1}{\sqrt{2}} \psi\left(A_{\widetilde{\Sigma}_i^{n,m}}, A_{\widetilde{T}_i^{n,m}}, A_{\widetilde{\Sigma}_j^{n,m}}, A_{\widetilde{T}_j^{n,m}}\right) &= \frac{1}{\sqrt{2}} d\left(\left[A_{\widetilde{\Sigma}_i^{n,m}}, A_{\widetilde{T}_j^{n,m}}\right], e\right) \\ &= \frac{1}{\sqrt{2}} d\left(A_{\left[\widetilde{\Sigma}_i^{n,m}, \widetilde{T}_j^{n,m}\right]}, A_e\right) \\ &= \sqrt{d\left(\left[\widetilde{\Sigma}_i^{n,m}, \widetilde{T}_j^{n,m}\right]\right)}. \end{aligned}$$

Thus, $\psi\left(A_{\widetilde{\Sigma}_i^{n,m}}, A_{\widetilde{T}_j^{n,m}}\right) = 0$ if $i < j$ and $\psi\left(A_{\widetilde{\Sigma}_i^{n,m}}, A_{\widetilde{T}_j^{n,m}}\right) \geq \sqrt{1-\eta} > 1 - \varepsilon$ if $i \geq j$. This shows that the sequence $\left(\left(A_{\widetilde{\Sigma}_i^{n,m}}, A_{\widetilde{T}_i^{n,m}}\right)\right)_{i=1}^n$ is a $\prec_{\frac{1}{\sqrt{2}}\psi}$-chain in U_m. Since this is true for every $\varepsilon > 0$, $n \in \mathbb{N}$ and $m > \frac{3^n}{\varepsilon}$, $(U_n)_{n\in\mathbb{N}}$ has the order property. ■

Teorema 7.7.4 *If $\neg CH$ holds, then there are $2^{\mathfrak{c}}$-many nonisomorphic universal sofic groups and universal hyperlinear groups.*

Bibliography

[B] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C*-algebras and von Neumann algebras, Operator Algebras and Noncommutative Geometry, III.

[BYBHU] I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model Theory with Applications to Algebra and Analysis, Vol. II (Z. Chatzidakis et al., eds.), Lecture Notes series of the London Math. Society., no. 350, Cambridge University Press, 2008, pp. 315-427

[D] A. Dow, *On ultrapowers of boolean algebras*, Topology Proc. 9 (1984), no. 2, 269-291

[FHS1] I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras I: Stability*, preprint, http://arxiv.org/abs/0908.2790, 2009.

[FHS2] I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras II: Model Theory*, preprint, http://arxiv.org/abs/0908.2790, 2009.

[FS] I. Farah and S. Shelah, *A dichotomy for the number of ultrapowers*, preprint, arXiv:0912.0406v1, 2009

[KP] A. Kwiatkowska, V. Pestov, *An introduction to hyperlinear and sofic groups*, arXiv:0911.4266v2 [math.GR], Lecture notes of the 7-th Appalachian set theory workshop (Cornell University, Ithaca, NY, November 22, 2008), 30 pp., to appear in the volume "Appalachian Set Theory 2006-2009" (James Cummings and Ernest Schimmerling, eds.), the Ontos-Verlag Mathematical Logic Series.

[P] V. Pestov, *Hyperlinear and sofic groups: a brief guide*, Bull. Symbolic Logic 14 (2008), 449-480.

[T] S. Thomas, *On the number of universal sofic groups*, Proceedings of the American Mathematical Society, Volume 138, Number 7, July 2010, Pages 2585–2590