# REPORT ON THE PROJECT: LIMIT LAWS FOR RANDOM WALKS AMONG CORRELATED RANDOM CONDUCTANCES

## 1. Purpose

The purpose of the visit was to start developing ideas concerning the reversible and "controllably" non-reversible random walks in random environment. The main standing question is for what processes in random environment can one prove a version of the (functional) central limit theorem under a diffusive scaling of the paths.

In the past several years, the host (Bálint Tóth) has put a lot of effort in relaxing the various *sector conditions* that have been used to handle non-symmetric part in the jump distribution; the guest (Marek Biskup) has in turn studied the validity of the *quenched invariance principles* for various strongly non-elliptic, albeit symmetric situations.

## 2. Results

The time spent on discussion was focused on the problem of a Markov chain whose generator has the following form:

$$L := S + A$$

where $S$ is the symmetric part and $A$ the antisymmetric part (as operators on a corresponding Hilbert space). Assuming that the setting is well posed — which requires natural assumptions about denseness of the domains, etc — the key question is under what condition can one guarantee that the skew-symmetric operator

$$T := S^{-1/2} A S^{-1/2}$$

is essentially self-adjoint. By earlier calculations due to the host, this would imply the existence of a martingale approximation on the domain of the operator $S^{-1/2}$ — a natural generalization of Kipnis-Varadhan theory.

The stumbling block is to check the validity of von Neuman's condition: $\mathrm{Ker}(T \pm 1) = \emptyset$. We focussed on the case $S := -\Delta$ and $A := F \cdot \nabla$, where $F$ is a bounded stationary vector field and $\nabla$ and $\Delta$ ar the usual gradient and Laplacian operators moved to the space of environments. It has been shown that the existence of a vector in the kernel would imply the existence of a function $U$ satisfying $\Delta U = F \cdot U$ which obeys a maximal principle, grows sublinearly but not-slower than with the square root of the distance. Unfortunately, we have not yet succeeded in ruling out the existence of this function altogether.

A second set of problems that have been discussed concerns random conductance problems where the conductances are derived from a Gaussian Free Field. For $(\phi_x)$ be a sample of this process over the hypercubic lattice (i.e., $x \in \mathbb{Z}^d$), one has a Markov chain on $\mathbb{Z}^d$ with transition probabilities

$$\mathsf{P}_\phi(x,y) := \frac{\mathrm{e}^{\alpha(\phi_y-\phi_x)}}{\mathscr{N}_\phi(x)}, \qquad |x-y|=1,$$

where

$$\mathscr{N}_\phi(x) := \sum_{y\colon\, |y-x|=1} \mathrm{e}^{\alpha(\phi_y-\phi_x)}$$

This walk exhibits trapping in $d=2$ and a annealed invariance principle is known in dimensions $d \geq 3$. However, it is not clear how to tackle the quenched invariance principle.

Two aspects of the problem have been discussed: Obviously, when the field is replaced by a massive field and the mass is taken to zero, one expects that the variance tends to zero in dimension $d=2$. This would mean that the two martingales in the standard Kipnis-Varadhan martingale approximation gradually cancel each other's contributions in this limit. It is rather mysterious why that happens. Second, one should be able to restrict the problem to a finite volume and then couple the field configuration to that with Dirichlet boundary condition. Then one could perhaps use all known information about the multifractal nature of the GFF to control the trapping time. This presumes the picture that the *geometric path* of the walk, stripped off the delays caused by trapping effects, is still of a Brownian nature. This aspect may need to be established first.

## 3. Summary and outlook

The visit was intended as an attempt to explore rather new and difficult areas and, as such, it has not yet produced definite marketable results. However, it has led to a number of interesting observations and a formulation of interesting directions of further research that could pave the way towards further understanding of random walks in random environment.