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Purpose of the visit

Background. Let (M, ω) be a symplectic manifold. A smooth map $s : (M, \omega) \to B$ is a *Lagrangian fibration* if $s^{-1}(b)$ is a Lagrangian submanifold of (M, ω) for each regular value $b \in B$. Such maps arise naturally in the study of completely integrable Hamiltonian systems via the Liouville-Mineur-Arnol'd theorem (cf. [Dui]), and in mirror symmetry via the SYZ conjecture (cf. [GS]). Examples range from toric manifolds to the well-known fibration of a K3 surface over $\mathbb{CP}^1$ with 24 singular fibres (cf. [Sym]). Henceforth, a Lagrangian fibration is assumed to be proper and have connected fibres unless otherwise stated.

Under these assumptions and in the absence of singular fibres, a Lagrangian fibration is a torus bundle and its topological and symplectic invariants are known (cf. [Dui]). The rigidity of Lagrangian bundles (*i.e.* fibrations with no singularities) is reflected in the following lemma (cf. [Dui]).

Lemma 1. *A manifold B is the base of a Lagrangian bundle if and only if it admits a smooth atlas $\mathcal{A}$ whose changes of coordinates are constant on connected components and are restrictions of* integral affine transformations *of $\mathbb{R}^n$, i.e. elements of*

$$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) := \mathrm{GL}(n; \mathbb{Z}) \ltimes \mathbb{R}^n.$$

A pair $(B, \mathcal{A})$ as in Lemma 1 is an *integral affine manifold* and the atlas $\mathcal{A}$ is an *integral affine structure*. Given $(B, \mathcal{A})$, the proof of Lemma 1 provides a technique to construct a *reference* Lagrangian bundle

$$\mathbb{T}^n \hookrightarrow (\mathrm{T}^*B/\Lambda_{\mathcal{A}}, \omega_0) \to B$$

which

i) induces the given integral affine structure on B;
ii) admits a globally defined Lagrangian section;
iii) provides the semiglobal symplectic model for any other Lagrangian bundle $(M, \omega) \to B$ which induces the atlas $\mathcal{A}$ on B, *i.e.* in a neighbourhood of each fibre of $(M, \omega) \to B$ there exists a local symplectomorphism $(M, \omega) \to (\mathrm{T}^*B/\Lambda_{\mathcal{A}}, \omega_0)$ which preserves the bundle structures.

The Lagrangian submanifold $\Lambda_{\mathcal{A}} \subset (\mathrm{T}^*B, \Omega)$ (where Ω denotes the canonical symplectic form on T^*B) is the total space of a $\mathbb{Z}^n$-bundle over B whose fibre is the integral span of the differentials of local integral affine coordinates given by $\mathcal{A}$; it is known in the literature as the *period lattice* associated to the integral affine manifold $(B, \mathcal{A})$ (cf. [Dui]). The topological (resp. symplectic) invariants of a Lagrangian bundle $(M, \omega) \to B$ inducing the atlas $\mathcal{A}$ on B can be interpreted as obstructions to the existence of a global topological (resp. symplectic) isomorphism between $(M, \omega) \to B$ and $(\mathrm{T}^*B/\Lambda_{\mathcal{A}}, \omega_0) \to B$ (cf. [Dui]). Therefore, any Lagrangian bundle over a given manifold B can be constructed by specifying the following data (cf. [DD, Dui, Sep]): –

A) an integral affine structure $\mathcal{A}$ on B;
B) a cohomology class $c \in \mathrm{H}^2(B; \mathcal{P})$, where $\mathcal{P}$ denotes the sheaf of sections of the period lattice $\Lambda_{\mathcal{A}}$. This is the topological invariant determining the isomorphism class of the bundle and it must satisfy an 'integrability' condition related to the obstruction to constructing a symplectic form on the total space of the corresponding torus bundle;
C) a cohomology class $\alpha \in \mathrm{H}^2(B; \mathbb{R})$ which gives the symplectic invariant of the bundle.

The above provides a solution to the *construction* problem of Lagrangian fibrations with no singularities. However, most Lagrangian fibrations arising from completely integrable Hamiltonian systems and mirror symmetry have singular fibres which are associated to interesting mathematical and physical phenomena (cf. [Dui, Gro1]). The focus of this report is on fibrations which occur frequently in integrable Hamiltonian systems and have so-called *non-degenerate* singularities. Such singularities decompose (topologically) as products of the following building blocks (cf. [Zun1]): –

- *elliptic* singularities (*e.g.* toric manifolds);
- *hyperbolic* singularities (*e.g.* mathematical pendulum);
- *focus-focus* singularities (*i.e.* the Lagrangian counterpart of nodal singularities in Lefschetz fibrations).

While [Zun2] provides a topological and symplectic classification theory for Lagrangian fibrations with such singularities which generalises [DD, Dui], there is no general method to construct Lagrangian fibrations over a given base manifold B and, therefore, there is a dearth of explicit examples (cf. [CBM1, GS]).

Objective. The purpose of the visit was to work on the construction problem of Lagrangian fibrations and, in particular, on understanding whether any Lagrangian fibration over a manifold B can be recovered by specifying data as in the case with no singularities. The data associated to a Lagrangian fibration over B necessarily include the following (note that this list may not be exhaustive): –

a) a *singular locus* $\Delta \subset B$, *i.e.* a closed subset which determines the singular values of the fibration;
b) an integral affine structure $\mathcal{A}_0$ on $B_0 = B \setminus \Delta$;
c) symplectic invariants defined near Δ which determine the 'fibred' symplectic topology near the singularities (cf. [CB, VN]);
d) appropriate generalisations of the topological and symplectic invariants of Lagrangian bundles to the case with singularities.

In the context of non-degenerate singularities, several aspects of the classification and construction problems have been studied in [LS, PVN, Sym, VN] under the restrictions that $\dim M = 4$ and in the absence of hyperbolic singularities. A crucial tool is the local integral affine structure of neighbourhoods of singular loci, which is determined by the non-degeneracy of the singularities (cf. [MZ]). The approach that this project takes is complementary to the one which underlies most of the literature in completely integrable Hamiltonian systems, since it assumes that there is no given Lagrangian fibration, but rather aims to construct such objects. In this sense, this project shares ideas with work on the SYZ conjecture carried out in [CBM1, GS, Gro2] and with work on constructing semitoric integrable systems (cf. [PVN]); however, seeing as the construction problem has not been studied in dimensions higher than 6, this research aims to expand and enrich known construction techniques to provide new examples of Lagrangian fibrations with many non-trivial topological and symplectic invariants. Moreover, it proposes to construct natural *reference* Lagrangian fibrations over B (once some extra data have been specified) which simplify the classification theory of [Zun2], as the topological and symplectic invariants defined therein are relative to a choice of a reference Lagrangian fibration.

Work carried out during the visit

The first step taken during the visit was to break the construction problem for Lagrangian fibrations down into smaller questions, which are listed below.

Step 1 Define an appropriate notion of integral affine manifolds with singularities which allows for a construction of a smooth *reference* Lagrangian fibration as in Lemma 1 above. This technique should allow to determine the topology of singular fibres of the reference Lagrangian fibration by considering the integral affine structure near the singular locus;

Step 2 Compactify the singular fibres using normal forms for non-degenerate singularities (cf. [MZ]). This involves studying a *gluing* problem whose solution will give rise to a generalisation of symplectic invariants in a neighbourhood of hyperbolic and focus-focus singularities (cf. [CB, DVN, VN]). Particular attention will be paid to gluing hyperbolic singularities in, as this case has not yet been considered in the existing literature; the techniques of *stitched* Lagrangian fibrations developed in [CBM2] are a starting point to solve this problem;

Step 3 Prove that *any* Lagrangian fibration with non-degenerate singularities can be constructed starting from the data determined in Step 1 and Step 2 above. The construction process is, in some sense, inverse to the classification theory of Lagrangian fibrations developed in [Zun2]; as such, the topological and symplectic invariants constructed in [Zun2] are going to correspond to the data needed to construct a Lagrangian fibration. Moreover, this approach will shed light on the reasons behind the assumptions used to develop the theory in [Zun2].

Step 4 Relate these construction techniques with other classification and construction results in the literature (e.g. [CBM1, GS, LS, PVN]).

During the visit, most of the work concentrated on solving Step 1 above; the results that were obtained are briefly described in the next section. It is important to remark that the techniques developed thus far only allow to deal with integral affine manifolds with singularities which arise from Lagrangian fibrations without hyperbolic blocks. The reason for this restriction is that if $s : (M, \omega) \to B$ is a Lagrangian fibration whose singularities have hyperbolic components, the singular locus Δ, *i.e.* the subset of B consisting of singular values of s, is locally given by the intersection of finitely many closed submanifolds of codimension 1. In particular, no point $x \in \Delta$ admits a neighbourhood U whose intersection with $B_0 = B \setminus \Delta$ is path-connected: this condition is key to the construction outlined below. However, it is expected that the results of next section can be extended to deal with locally codimension 1 singular loci by studying how to glue the structures constructed below along these subsets (cf. [CBM2]).

Description of the main results obtained

In this section the problem of defining a notion of integral affine manifold with singularities which allows to construct reference Lagrangian fibrations as in Lemma 1 is briefly discussed. Such a notion is suggested in Definition 4 and is used to prove Theorem 1, whose proof is only sketched here. It is important to notice that what is presented below is still work in progress.

There are various definitions of integral affine manifolds with singularities in the literature (cf. [CBM1, GS]) and the one given below is an adaptation of some of these to the purposes of this report.

Definition 1. An *integral affine manifold with singularities* is a triple $(B, \Delta, \mathcal{A}_0)$, where B is a smooth manifold, $\Delta \subset B$ is closed and locally a union of finitely many locally closed submanifolds of codimension at least 2, and $\mathcal{A}_0$ is an integral affine structure on the complement $B_0 = B \setminus \Delta$. The subset Δ is called the *singular locus* of $(B, \Delta, \mathcal{A}_0)$.

Remark 1. If $s : (M, \omega) \to B$ is a proper Lagrangian fibration with connected fibres and set of singular values Δ, then the proof of the Liouville-Mineur-Arnol‘d theorem in [Dui] implies that $B_0 = B \setminus \Delta$ inherits an integral affine structure $\mathcal{A}_0$. Therefore the triple $(B, \Delta, \mathcal{A}_0)$ is an integral affine manifold with singularities as defined above, provided Δ satisfies the condition of Definition 1.

The above definition of integral affine manifolds with singularities does not give sufficient control of the behaviour of integral affine structures near the singular loci to obtain the result of Theorem 1. Therefore, further conditions are needed; before stating these conditions, it is necessary to recall some results from integral affine geometry, which are only stated here (cf. [GH, GS]).

Definition 2. Let $(B, \mathcal{A})$ be an integral affine manifold. A (possibly locally defined) smooth function $f : B \to \mathbb{R}$ is said to be *integral affine* if in local integral affine coordinates $a^1, \ldots, a^n$ it is given by

$$f(a^1, \ldots, a^n) = \sum_{i=1}^{n} k_i a^i + c,$$

where $k_i \in \mathbb{Z}$ for $i = 1, \ldots, n$ and $c \in \mathbb{R}$ is a constant.

Proposition 1. *To an n-dimensional integral affine manifold $(B, \mathcal{A})$ and a point $b \in B$ can be associated a representation*

$$\mathfrak{a} : \pi_1(B; b) \to \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$$

called the affine holonomy *of $(B, \mathcal{A})$. The composite*

$$\mathfrak{l} := \mathrm{Lin} \circ \mathfrak{a},$$

where $\mathrm{Lin} : \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) \to \mathrm{GL}(n; \mathbb{Z})$ *is the natural projection, is called the* linear holonomy *of $(B, \mathcal{A})$. The obstruction for $\mathfrak{a}$ to admit a fixed point is a cohomology class*

$$r_{(B,\mathcal{A}),b} \in \mathrm{H}^1(\pi_1(B); \mathbb{R}^n_{\mathfrak{l}}) \cong \mathrm{H}^1(B; \mathbb{R}^n_{\mathfrak{l}})$$

called the radiance obstruction *of $(B, \mathcal{A})$ (with respect to a choice of base point $b \in B$).*

Definition 3. An integral affine manifold $(B, \mathcal{A})$ whose radiance obstruction $r_{(B,\mathcal{A}),b}$ vanishes is said to be *radiant.*

Remark 2. The notion of a radiant integral affine manifold is independent of the choice of base point $b \in B$ needed to construct the affine and linear holonomies.

With Definitions 2 and 3 in hand, the notion of *suitable* integral affine manifolds with singularities is suggested below, inspired by some results in [GS].

Definition 4. An integral affine manifold with singularities $(B, \Delta, \mathcal{A}_0)$ is *suitable* if every $x \in \Delta$ admits an open, contractible neighbourhood $U \subset B$ satisfying the following conditions

1) every locally defined integral affine function
$$f : \underbrace{U \setminus \Delta}_{:=U_0} \to \mathbb{R}$$
admits a smooth extension to U;
2) the integral affine manifold $(U_0, \mathcal{A}_0|_{U_0})$ is radiant.

Remark 3. To the best of the author's knowledge, almost all integral affine manifolds with singularities arising in the literature regarding Lagrangian fibrations are suitable (cf. [CBM1, GS, PVN]). Moreover, [GS] gives a method to construct examples of such manifolds, with the difference that B is only assumed to be a topological manifold.

Suitable integral affine manifolds with singularities admit *period lattices* as in the regular case and, therefore, it is possible to construct reference Lagrangian fibrations over such manifolds. This is the content of the following theorem, which is the main result obtained during the visit.

Theorem 1. *Let $(B, \Delta, \mathcal{A}_0)$ be a suitable integral affine manifold with singularities. There exists a smooth closed Lagrangian submanifold $\Lambda \subset (\mathrm{T}^*B, \Omega)$ called the* period lattice *associated to $(B, \Delta, \mathcal{A}_0)$. The quotient T^*B/Λ inherits a symplectic form ω_0 and there exists a Lagrangian fibration*
$$(\mathrm{T}^*B/\Lambda, \omega) \to B$$
which admits a globally defined Lagrangian section.

Sketch of proof. Given a suitable integral affine manifold with singularities $(B, \Delta, \mathcal{A}_0)$, consider the period lattice $\Lambda_0 \subset (\mathrm{T}^*B_0, \Omega_0)$ associated to the integral affine manifold $(B_0, \mathcal{A}_0)$. This is the total space of a $\mathbb{Z}^n$-bundle over B_0 which is spanned by locally exact forms; therefore, Λ_0 is a Lagrangian submanifold of $(\mathrm{T}^*B_0, \Omega_0)$. The idea is to extend Λ_0 smoothly over the singular locus Δ so as to define a Lagrangian submanifold of (T^*B, Ω).

Let $\mathrm{pr}_0 : \Lambda_0 \to B_0$ denote the projection and, for $x \in B_0$, let $\Lambda_{0,x} = \mathrm{pr}_0^{-1}(x)$. Note that for every point $x \in B_0$ and any element $\sigma_x \in \Lambda_{0,x}$ there exists an open neighbourhood $U \subset B_0$ and a locally defined smooth section $\sigma_U : U \to \mathrm{pr}_0^{-1}(U)$ with $\sigma_U(x) = \alpha_x$. Existence of such sections allows to prove that Λ_0 is a closed submanifold of T^*B and that the quotient $\mathrm{T}^*B_0/\Lambda_0$ obtained from the fibrewise action defined by translating along elements of Λ_0 is a smooth manifold (cf. [Vai]). If these last two properties hold for Λ, then T^*B/Λ is a manifold. Therefore, it suffices to define Λ so that the existence of local sections as above holds to endow T^*B/Λ with a smooth structure.

Let $x \in B_0$. There exists a coordinate neighbourhood $U \subset B_0$ with local integral affine coordinates $a^1_U, \ldots, a^n_U$. Define

$$\Lambda|_U = \{(y, \mathbf{p}) \in \mathrm{T}^*U : \mathbf{p} \in \mathbb{Z}\langle \mathrm{d}a^1_U, \ldots, \mathrm{d}a^n_U \rangle\} (= \Lambda_0|_U). \tag{1}$$

As in the case with no singularities, this definition is independent of the choice of integral affine coordinates near x. It now remains to define Λ near the singular locus Δ. Let $x \in \Delta$ and let $U \subset B$ be an open neighbourhood of x as in Definition 4. Set $U_0 = U \setminus \Delta$ and denote the sheaf of sections of $\Lambda_0|_{U_0}$ by $\mathcal{P}_0$. By the local structure of Δ specified in Definition 1, U_0 is path-connected. Consider $\mathrm{H}^0(U_0; \mathcal{P}_0)$, the group of global (over U_0) sections of $\mathcal{P}_0$. The sheaf $\mathcal{P}_0$ is a local system of coefficients with stalk isomorphic to $\mathbb{Z}^n$ and twisted by a representation
$$\rho_0 : \pi_1(U_0; y) \to \mathrm{GL}(n; \mathbb{Z}),$$

where $y \in U_0$ and ρ_0 is the inverse transposed of the linear holonomy of the integral affine manifold $(U_0, \mathcal{A}_0|_{U_0})$ with respect to the base point y. Then $\mathrm{H}^0(U_0; \mathcal{P}_0)$ is the subgroup of $\mathbb{Z}^n$ which is invariant under the local monodromy action of $\pi_1(U_0)$ defined above. Since sections of Λ_0 are locally exact forms, elements of $\mathrm{H}^0(U_0; \mathcal{P}_0)$ are closed 1-forms defined on U_0. The aim is to show that these extend to 1-forms defined over U.

Let $q_0 : \tilde{U}_0 \to U_0$ denote the universal covering and fix $\theta \in \mathrm{H}^0(U_0; \mathcal{P}_0)$. Set $\tilde{\theta} = q_0^* \theta$. Since $\tilde{U}_0$ is simply connected, there exists a function $f : \tilde{U}_0 \to \mathbb{R}$ such that

$$\tilde{\theta} = \mathrm{d}f.$$

Since $\tilde{\theta} \in q_0^* \Lambda_0|_{U_0}$, it follows that f can be chosen so that it is an integral affine function on $\tilde{U}_0$. In fact, this function is invariant under the action of $\pi_1(U_0)$ on $\tilde{U}_0$ by deck transformations; this fact follows from the fact that $(U_0, \mathcal{A}_0|_{U_0})$ is radiant or, equivalently, that the affine holonomy has no translational component. Therefore, f descends to an integral affine function $h_0 : U_0 \to \mathbb{R}$. Note that $q_0^* \mathrm{d}h_0 = \tilde{\theta}$; since q_0 is a local diffeomorphism, q_0^* is injective, which implies that $\theta = \mathrm{d}h_0$, thus proving that θ is an exact 1-form whose potential can be chosen to be an integral affine function on U_0. Since θ is arbitrary, it follows that all global sections of $\mathcal{P}_o$ are the differential of some integral affine function on U_0.

Thus set

$$\mathrm{H}^0(U_0; \mathcal{P}_0) = \mathbb{Z}\langle \mathrm{d}h_0^1, \ldots, \mathrm{d}h_0^k \rangle,$$

for some integral affine functions $h_0^1, \ldots, h_0^k$. Since $(B, \Delta, \mathcal{A}_0)$ is suitable, the integral affine functions $h_0^1, \ldots, h_0^k$ admit unique smooth extensions $h^1, \ldots, h^n : U \to \mathbb{R}$. The exact 1-forms $\mathrm{d}h^1, \ldots, \mathrm{d}h^k$ defined on U extend $\mathrm{d}h_0^1, \ldots, \mathrm{d}h_0^k$. By shrinking U if needed, it may be assumed that U is a coordinate neighbourhood for B. Define

$$\Lambda|_U := \{(y, \mathbf{p}) \in \mathrm{T}^*U : \mathbf{p} \in \mathbb{Z}\langle \mathrm{d}h^1, \ldots, \mathrm{d}h^k \rangle\}. \tag{2}$$

Since $h^1, \ldots, h^k$ restrict to integral affine functions on U_0, it follows the definitions of Λ given by equations (1) and (2) agree on U_0. This shows that Λ is well-defined.

As above, let $\mathrm{pr} : \Lambda \to B$ be the natural projection and, for $x \in B$, let $\Lambda_x := \mathrm{pr}^{-1}(x)$. The definition of Λ implies that for any $x \in B$ and any element $\sigma_x \in \Lambda_x$ there exists a local section $\sigma_U : U \to \mathrm{pr}^{-1}(U)$ with $\sigma_U(x) = \alpha_x$. Using the same argument as in the regular case (cf. [Vai]), it follows that Λ is a closed submanifold of T^*B and that T^*B/Λ is a smooth manifold. Since local sections of Λ are closed 1-forms, the fibrewise action defined by translating along Λ is by symplectomorphisms of (T^*B, Ω). Therefore, T^*B/Λ inherits a symplectic form ω_0 which makes the fibration $(\mathrm{T}^*B/\Lambda, \omega_0) \to B$ Lagrangian. The zero section $z : B \to \mathrm{T}^*B$ descends to a Lagrangian section σ of $(\mathrm{T}^*B/\Lambda, \omega_0) \to B$ and the result of the theorem follows. □

Future collaboration with host institution

The work carried out during the visit was simply the first step in a much larger project that aims at understanding completely integrable Hamiltonian systems and generalisations thereof (including non-commutative integrable systems on Poisson manifolds). Thus there will certainly be future collaboration with Rui Loja Fernandes, Miguel Abreu and other people at Istituto Superior Técnico in Lisbon.

Projected publications

Below are listed the two projected publications arising from the work carried out during the visit in Lisbon.

1) Sepe, D. *The integral affine geometry of isotropic bundles*;
2) Sepe, D. *Construction of Lagrangian fibrations.*

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