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1. Purpose of the visit

Spaces of directed paths. M. Raussen provided models ([2], [3]) of spaces of directed paths on cubical complexes, while I constructed [5] the similar model for directed simplicial complexes. We want to find a generalization of these results for a larger class of directed spaces, like prod-simplicial complexes or directed realizations of simplicial sets. Another concept is to apply these constructions for calculations of certain invariants, like the component category.

Categories of directed spaces. In my recent paper [6] I constructed a common subcategory of the categories of d-spaces and streams and proved that it has good categorical properties (i.e. it is complete, cocomplete and Cartesian closed). Moreover, I proved that we can put some additional restrictions on good d-spaces and obtain the category which is still complete, cocomplete an Cartesian closed. We want to discuss possible applications of these results, or to improve them, i.e. to find a better category of directed spaces.

2. Description of the results

The results stated below were obtained as a result of cooperation of Marin Raussen, Lisbeth Fajstrup and me during my stay in Aalborg (January 15-21, 2012).

2.1. **Revision of the paper** [5]**.** I have received many suggestions from Martin Raussen concerning my paper [5], which helped me to make it more understandable. After these corrections, the paper has been accepted for publication. A suitable remark has been placed in this paper.

2.2. **Tameness of paths in cubical complexes.** A directed path in a d-simplicial complex is *tame* iff every its segment either contains a vertex, or is contained in a simplex. In other words, a d-path is tame if it moves from one simplex into another at vertices only. The main technical result of [6] states that the subspace of tame directed paths is a deformation retract of the space of all directed paths (with end-points fixed in both cases). We have proved that a similar result (with a similar notion of a tame path) holds for cubical complexes; it is even stronger since we do not need to assume that complexes we consider contain no loops.

2.3. **Path spaces in cubical complexes.** In [5], I have constructed a cubical complex, called trace complex, which models a space of paths of directed spaces in d-simplicial complexes. The tameness result, mentioned in the previous paragraph, allowed us to produce a similar construction for cubical complexes. The resulting analogue of a trace complex has a structure of "permutohedral complex" — an analogue of simplicial and cubical complexes, where the cells are products of permutohedra (instead of simplices or cubes).

2.4. **Directed paths in tori with holes.** In a sequence of papers ([2], [3],[4]) Martin Raussen provided effective methods of calculating a homotopy type of a space of directed paths in n-dimensional cube, with an arbitrary collection of rectangular areas removed. We tried to obtain similar results for tori (again, with possibly some rectangular areas removed). The satisfactory answer in a general case seems to be hard to obtain; however we made a substantial progress in the case where only one rectangular area is removed from a cube. This problem is equivalent to calculating spaces of directed loops on skeleta of tori (presented as cubical complexes in a natural way). Namely,

- We have calculated a homotopy type of the space of directed loops on the 2-skeleton of 3-dimensional torus. This space is a disjoint sum of components corresponding to positive elements of the fundamental group of the 3-torus. We proved that every component is aspherical, and found a presentation of its fundamental group in a generators-and-relations form. Furthermore, we have constructed a finite CW-complex which is homotopy equivalent to this space, and allows to calculate its homology groups.
- In the case of 2-skeleton of an arbitrary torus, a similar construction seems to lead to analogous results. We think we can prove it but there are still many details which need to be checked.
- We hope that techniques we developed can provide an answer in the case of arbitrary skeleton of a torus. The directed loop space is no longer an Eilenberg-McLane space, but probably we will be able to present this space (up to homotopy) as a finite CW-complex and calculate its homology groups.

2.5. **ldpc-spaces.** I have discussed with Lisbeth Fajstrup the relationship between ldpc-spaces [6] and cube generated spaces [1], and compared advantages and disadvantages of using these classes of objects in other applications. We have also constructed examples of d-spaces which are contained in one of these classes but not in the other one. It proves that none of these classes contain the other one.

2.6. **Categories of directed spaces.** On January 19, I gave a talk at Aalborg University entitled "Categories of directed spaces". It was a presentation of my paper [6].

3. Future collaboration and expected publications

Results described in paragraphs 2.2 and 2.3 can probably be applied in calculations of directed path spaces, and, if it happens, we plan to publish it as a part of some article. We are going to complete results described in paragraph 2.4, and then to write and publish an article containing them.

References

[1] L. Fajstrup, J. Rosický, *A convenient category for directed homotopy*, Theory and Applications of Categories, **21** (2008), 720.
[2] M. Raussen, *Trace spaces in a pre-cubical complex*, Topology Appl. **156** (2009), 1718-1728.
[3] M. Raussen, *Simplicial models for trace spaces*, Algebr. Geom. Topol. **10** (2010), 16831714.
[4] M. Raussen, *Execution spaces for simple higher dimensional Automata*, Research report R-2010-14, Department of Mathematical Sciences, Aalborg University; accepted for publication in Appl. Algebra Engrg. Comm. Comput.
[5] K. Ziemiański, *Cubical model for path spaces in d-simplicial complexes*, `http://dx.doi.org/10.1016/j.topol.2012.02.005`, to appear in Topology and Applications.
[6] K. Ziemiański, *Categories of directed spaces*, preprint, 2011, to appear in Fundamenta Mathematicae.