SCIENTIFIC REPORT

MICHAEL BRANDENBURSKY

1. Purpose of the visit

The purpose of the visit was to continue a scientific collaboration with Jarek Kedra, which has started in 2011 and resulted in two published papers, see [1, 2].

2. Description of the work and main results

Let $\mathbf{G} := \mathrm{Diff}^\infty(\mathbf{D}, \partial\mathbf{D}, \mathrm{area})$ be the group of C^∞ area-preserving diffeomorphisms of the unit disc $\mathbf{D}$ in the plane, which are identity near the boundary $\partial\mathbf{D}$. It is a known fact that for every $g \in \mathbf{G}$ there exists $k \in \mathbf{N}$ such that $g = h_1 \circ \ldots \circ h_k$, where h_i, here $1 \geq i \leq k$, is a time-one flow generated by some C^∞ function $H_i \colon \mathbf{D} \to \mathbf{R}$, i.e. each h_i is an autonomous diffeomorphism. Let us define a bi-invariant norm $\|\cdot\|_{\mathrm{Aut}}$ on $\mathbf{G}$ as follows: $\|g\|_{\mathrm{Aut}}$ is a minimal number k, such that $g = h_1 \circ \ldots \circ h_k$, where each h_i is an autonomous diffeomorphism. This norm induces a bi-invariant metric on $\mathbf{G}$, i.e. $\mathbf{d}_{\mathrm{Aut}}(f, g) := \|fg^{-1}\|_{\mathrm{Aut}}$.

Recall that a homogeneous *quasi-morphism* on a group K is a function $\varphi \colon K \to \mathbf{R}$ which satisfies the homomorphism equation up to a bounded error: there exists $D_\varphi \geq 0$ such that $|\varphi(ab) - \varphi(a) - \varphi(b)| \leq D_\varphi$ for all $a, b \in K$, and in addition for every $n \in \mathbf{Z}$ we have $\varphi(a^n) = n\varphi(a)$. Denote by $Q(\mathbf{G}, \mathrm{Aut})$ the space of homogeneous quasi-morphisms on $\mathbf{G}$ that are identically zero on all autonomous diffeomorphisms. Our main results include the following:

Theorem 1. *The vector space* $Q(\mathbf{G}, \mathrm{Aut})$ *is infinite-dimensional.*

As a corollary we obtain

Corollary. The metric space $(\mathbf{G}, \mathbf{d}_{\mathrm{Aut}})$ has an infinite diameter.

A similar result to ours for the group of area-preserving diffeomorphisms of a 2-sphere $\mathbf{S}^2$ was proved by Gambaudo-Ghys in [3]. Before proceeding we need the following

Definition. *Let $(K, \mathbf{d}_K)$ and $(K', \mathbf{d}_{K'})$ be two metric groups. A function $f\colon K \to K'$ is a bi-Lipschitz embedding if it is an injective homomorphism, and there exists a constant $C \geq 1$ such that*

$$C^{-1}\mathbf{d}_K(g,h) \leq \mathbf{d}_{K'}(f(g), f(h)) \leq C\mathbf{d}_K(g,h).$$

Let $\mathbf{Z}^n$ be a free Abelian group of rank n. We equip it with a word metric, which of course is bi-invariant. Now we are ready to present our main theorem.

Theorem 2. *For every $n \in \mathbf{N}$ the group $(\mathbf{G}, \mathbf{d}_{\mathrm{Aut}})$ contains bi-Lipschitz embedded $\mathbf{Z}^n$.*

The group $\mathbf{G}$ may be equipped with the famous Hofer metric [4, 5]. Similar results to ours with respect to the Hofer metric were obtained by Py and Usher, see [6, 7]. In addition, in the upcoming paper we plan to discuss the relation between this bi-invariant metric, the Hofer metric and the fragmentation metric.

3. Projected publications

We (Brandenbursky and Kedra) have started to write the paper "The autonomous metric on the group of area-preserving diffeomorphisms of the 2-disc", which will include the results discussed above and which were obtained during a visit of Michael Brandenbursky to Aberdeen. ESF will be acknowledged in this paper.

References

[1] Brandenbursky M.: *Quasi-morphisms and L^p-metrics on groups of volume-preserving diffeomorphisms,* to appear in the Journal of Topology and Analysis.

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[4] Hofer H.: *On the topological properties of symplectic maps,* Proc. Roy. Soc. Edinburgh Sect. A 115 (1990), no. 1-2, 25–38.

[5] Lalonde F., McDuff D.: *The geometry of symplectic energy,* Ann. of Math. (2) 141 (1995), no. 2, 349–371.

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[7] Usher M.: *Hofer's metric and boundary depth*, ArXiv:11074599, 2011.

Department of Mathematics, Vanderbilt University, Nashville, TN
E-mail address: `michael.brandenbursky@vanderbilt.edu`