SHORT REPORT FOR INFTY SHORT VISIT NO. 4955

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Overview

The main aim of the visit was participation in *Thematic Program on Forcing and its Applications* semester at Fields Institute and cooperation with other participating mathematicians. I have spent at Fields Institute one month. Two weeks of this period were supported by the INFTY grant, the rest by Fields Institute. I participated in the workshop *Forcing Axioms and their Applications*, in set theory seminars and in regular lectures given by Alan Dow. I cooperated mainly with David Chodounsky and with Jan Pachl and I enclose below the detailed report on our scientific activities.

I would like to thank David and Jan for nice cooperation and Dana Bartošová for agreeing for being my host in Toronto.

Counterexamples in towers and gaps.

My main collaborator in Toronto was David Chodounsky (Fields Institute). We were working on project *Counterexamples in towers and gaps*.

We call *a tower* a family of ω_1 many subsets of ω well-ordered by $\subseteq^*$ (almost inclusion). For a given tower $\mathcal{T}$ we study properties of the order $(\mathcal{T}, \subseteq)$ (note, that here we mean the real inclusion). The first dividing line here is the property of having an uncountable anti-chain. There are towers whose every sub-tower (uncountable subfamily) has an uncountable $\subseteq$-antichain (we call them *special* towers) and consistently (under $\mathsf{MA}(\omega_1)$) all towers are like that. On the other hand if $\mathfrak{b} = \omega_1$, then there is a tower without an uncountable $\subseteq$-antichain.

In the realm of special towers we can distinguish one more Specie of a tower. We say that a tower $(T_\alpha)_{\alpha\in\omega_1}$ is *Hausdorff* if $\{\xi < \alpha \colon T_\xi \setminus T_\alpha \subseteq n\}$ is finite for each $\alpha \in \omega_1$ and each natural n. Every Hausdorff tower is special and the property of being Hausdorff seems to be more global than the property of being special: contrary to the case of special towers every tower generating an ideal generated by a Hausdorff tower is Hausdorff.

We studied properties of those objects. The project started in November 2011 and before the visit we already proved several results in this subject. Among other things, the analysis of different kinds of towers has brought a simple proof of the fact that there are indestructible gaps not equivalent to a Hausdorff gap (the first example of this sort, but of different nature, was found by James Hirschorn, [2]).

During the visit we managed to prove the following facts and theorems:

- Under PID the following equivalence is true: there is a non-special tower if and only if $\mathfrak{b} = \omega_1$;
- It is consistent that there is a non-special tower and $\mathfrak{b} > \omega_1$;

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- Under $\mathsf{MA}(\omega_1)$ all towers have the following property: it is enough to modify each level of a tower by adding certain point and removing certain point to obtain a tower which forms a $\subseteq$-antichain;
- Each tower whose orthogonal is not generated by a single set is a half of a gap, but not necessarily of (ω_1, ω_1)-gap.

Some part of my visit I devoted to the question if every tower either generates an ideal generated by a Hausdorff tower (and thus is Hausdorff itself) or it generates an ideal generated by a non-special tower. We suspected that the tower being a half of the gap constructed by Hirschorn can be a counterexample for the above statement. However, we were unable to verify it.

Anyway, the project seems to be almost finished and we expect that we will submit a publication at the beginning of 2013.

Images of regularly monocompact measures.

I worked with Jan Pachl (Fields Institute) on the problems connected to the notion of regularly monocompact measures.

David Fremlin asked if every measure space whose measurable sets form a sub-σ-algebra of $\mathsf{Borel}[0,1]$ is countably compact (see [1] for the terminology). In [1] we provided a partial answer to that question. Namely, we showed that every such a measure space is an image of a regularly monocompact measure space. In [3] Jan Pachl proved that if a measure space is an image of countably compact measure space, then it is countably compact itself.

We tried to prove that each measure space being an image of regularly monocompact measure space is regularly monocompact. It would strengthen a little bit the theorem from [1].

We haven't been able to prove anything concrete, but some ideas which appeared during the discussions sound promising.

References

[1] P. Borodulin-Nadzieja, G. Plebanek, *On compactness of measures on Polish spaces*, Illinois Journal of Mathematics, Volume 49, Issue 2 (2005), 531-545;
[2] J. Hirschorn, *On the strength of Hausdorff's gap condition*, preprint;
[3] J. Pachl, *Disintegration and compact measures*, Math. Scand. 43 (1978/79), no. 1, 157-168.